Periodicity in Quantum Calculus

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Definition 1.1

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$$q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\} = \{1, q, q^2, q^3, q^4, \ldots \},$$

is called the quantum time scale.
Definition 1.2

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, define the forward jump operator $\sigma(t) = \inf \{ s \in \mathbb{T} : s > t \}$, and the backward jump operator $\rho(t) = \sup \{ s \in \mathbb{T} : s < t \}$. Here, let $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

Define the graniness function $\mu(t) = \sigma(t) - t$. 

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Definition 1.2

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Define the graniness function $\mu : \mathbb{T} \to [0, \infty)$ by

$$\mu(t) = \sigma(t) - t.$$
• On $\mathbb{T} = \mathbb{Z}$, $\sigma(t) = t + 1$, $\rho(t) = t - 1$ and $\mu(t) \equiv 1$. 

• On $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$, $\rho(t) = t$ and $\mu(t) \equiv 0$.

• On $\mathbb{T} = q\mathbb{N}_0$, $\sigma(t) = qt$, $\rho(t) = t + (t > 1)$, and $\mu(t) = (q-1)t$. 

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• On $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$, $\rho(t) = t$ and $\mu(t) \equiv 0$.
• On $T = q^{\mathbb{N}_0}$, $\sigma(t) = qt$, $\rho(t) = \frac{t}{q}$ ($t > 1$), and $\mu(t) = (q - 1)t$. 
Definition 1.3

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- If $\rho(t) < t < \sigma(t)$, $t$ is isolated.
- If $\rho(t) = t = \sigma(t)$, $t$ is dense.
Consider the time scale $\mathbb{T}$ given by

\[ \mathbb{T} \text{ given by}
\begin{align*}
\text{• } & t \text{ is left-dense and right-scattered.} \\
\text{• } & x \text{ is isolated.} \\
\text{• } & y \text{ is left-scattered and right-dense.} \\
\text{• } & z \text{ is dense.}
\end{align*} \]
Consider the time scale $\mathbb{T}$ given by

- $t$ is left-dense and right-scattered.
- $x$ is isolated.
- $y$ is left-scattered and right-dense.
- $z$ is dense.
Definition 1.4

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T} \setminus \text{sup} \mathbb{T}$. Then if $f$ is continuous at $t$ and $t$ is right-scattered ($t < \sigma(t)$), then $f$ is differentiable at $t$ with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Notice this is the slope of the secant line connecting $(t, f(t))$ and $(\sigma(t), f(\sigma(t)))$. 
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• On $\mathbb{T} = \mathbb{N}$,

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• On $\mathbb{T} = q^{\mathbb{N}_0}$,

$$f^\Delta(t) = D_q f(t) = \frac{f(qt) - f(t)}{(q - 1)t}.$$
Definition 1.5

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T} \setminus \text{sup} \mathbb{T}$. Then if $f$ is continuous at $t$ and $t$ is right-dense ($t = \sigma(t)$), then

$$f(t) = \lim_{s \to t} f(t) - f(s),$$

provided the limit exists. Notice the similarity between $f(t)$ and $f'(t)$. 

• On $\mathbb{T} = \mathbb{R}$, $f(t) = f'(t)$.
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\[
f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},
\]

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Notice the similarity between \( f^\Delta \) and \( f' \).

- On \( \mathbb{T} = \mathbb{R} \),

\[
f^\Delta(t) = f'(t)
\]
Theorem 1.6

Assume \( f, g : \mathbb{T} \rightarrow \mathbb{R} \) are differentiable at \( t \). Then

\[
(f + g)\Delta(t) = f\Delta(t) + g\Delta(t),
\]

Here \( f\sigma(t) = f(\sigma(t)) \).
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(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t),
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and

$$\left( \frac{f}{g} \right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}. $$
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$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$ 

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On $\mathbb{T} = \mathbb{R}$, the function $y(t) = e^{at}$ is uniquely determined as the solution of the initial value problem

$$y' = ay, \ y(0) = 1.$$
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Similarly, on a general time scale $\mathbb{T}$, we define $e_a(t, t_0)$ as the unique solution of the initial value problem

$$y^\Delta = ay, \ y(t_0) = 1.$$
Let’s find $e_1(t,0)$ on $\mathbb{T} = \mathbb{Z}$. 
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Let’s find $e_1(t, 0)$ on $\mathbb{T} = \mathbb{Z}$. Try a guess of $e_1(t, 0) = a^t$ for some $a$. Then

$$e_1^\Delta(t, 0) = a^{t+1} - a^t = a^t(a - 1) = a^t = e_1(t, 0)$$

iff $a - 1 = 1$, or $a = 2$. 
Let’s find $e_1(t,0)$ on $\mathbb{T} = \mathbb{Z}$. Try a guess of $e_1(t,0) = a^t$ for some $a$. Then

$$e_1^\Delta(t,0) = a^{t+1} - a^t = a^t(a - 1) = a^t = e_1(t,0)$$

iff $a - 1 = 1$, or $a = 2$. So $e_1(t,0) = 2^t$ on $\mathbb{T} = \mathbb{N}$. 
• On $2^\mathbb{N}_0$,

$$e_1(t, 1) = \prod_{s \in [1, t) \cap \mathbb{T}} (1 + s) = 2 \cdot 3 \cdot 5 \cdots \left(\frac{t}{2} + 1\right).$$
• On $2^{\mathbb{N}_0}$,

$$e_1(t, 1) = \prod_{s \in [1, t) \cap \mathbb{T}} (1 + s) = 2 \cdot 3 \cdot 5 \cdots \left(\frac{t}{2} + 1\right).$$

So, for example,

$$e_1(2, 1) = 2, \quad e_1(4, 1) = 2 \cdot 3 = 6, \quad e_1(8, 1) = 2 \cdot 3 \cdot 5 = 30.$$
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Notice

$$e_1^\Delta(2, 1) = \frac{6 - 2}{2} = 2 = e_1(2, 1),$$
• On $2^\mathbb{N}_0$,

$$e_1(t, 1) = \prod_{s \in [1, t) \cap \mathbb{T}} (1 + s) = 2 \cdot 3 \cdot 5 \cdots \left(\frac{t}{2} + 1\right).$$

So, for example,

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Notice

$$e_1^\Delta(2, 1) = \frac{6 - 2}{2} = 2 = e_1(2, 1),$$

and

$$e_1^\Delta(4, 1) = \frac{30 - 6}{4} = 6 = e_1(4, 1).$$
Definition 1.7

If $F^\Delta(t) = f(t)$, define the indefinite integral of $f$ by

$$\int f(t) \Delta t = F(t) + C,$$

and the definite integral of $f$ from $a$ to $b$ by

$$\int_a^b f(t) \Delta t = F(b) - F(a).$$
• If $T = \mathbb{R}$,
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\[
\int_a^b f(t) \Delta t = \int_a^b f(t) dt.
\]
• If $T = \mathbb{R}$,

$$\int_{a}^{b} f(t) \Delta t = \int_{a}^{b} f(t) dt.$$

• If $T = \mathbb{N}$,
\begin{itemize}
  \item If $T = \mathbb{R}$,
    \[
    \int_{a}^{b} f(t) \Delta t = \int_{a}^{b} f(t) dt.
    \]
  \item If $T = \mathbb{N}$,
    \[
    \int_{a}^{b} f(t) \Delta t = \sum_{t=a}^{b-1} f(t).
    \]
\end{itemize}
• If $T = q^{N_0}$,
• If $\mathbb{T} = q^\mathbb{N}_0$,

$$
\int_{q^m}^{q^n} f(s) \Delta s = \int_{q^m}^{q^n} f(s) dq_s = (q - 1) \sum_{k=m}^{n-1} q^k f(q^k). \quad (1)
$$
Definition 2.1

A time scale, denoted by $\mathbb{T}$, is said to be additively $T$-periodic if there exists a $T > 0$ such that $t \pm T \in \mathbb{T}$ for all $t \in \mathbb{T}$ (see [3]).
Definition 2.1

A time scale, denoted by \( \mathbb{T} \), is said to be additively \( T \)-periodic if there exists a \( T > 0 \) such that \( t \pm T \in \mathbb{T} \) for all \( t \in \mathbb{T} \) (see [3]). A function \( f \) on additively periodic domain \( \mathbb{T} \) with period \( T \) is said to be periodic with period \( P \) if there exists an \( n \in \mathbb{N} \) such that \( P = nT \), \( f(t \pm P) = f(t) \) for all \( t \in \mathbb{T} \), and \( P \) is the smallest number such that \( f(t \pm P) = f(t) \).
Notice, since the time scale $q^\mathbb{N}_0$ is not additive, we cannot define periodicity on $q^\mathbb{N}_0$ in a same way we do on additively periodic domains. Recently, two different definitions of periodicity on $q^\mathbb{N}_0$ have arose. We wish to show a relationship between these two definitions of periodicity. We will then show the existence of $P$-periodic solutions of a $q$-Volterra integral equation.
Definition 2.2 (Bohner and Chieochan, [2])

Let $P \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is said to be $P$-periodic if

$$f(t) = q^P f \left( q^P t \right) \text{ for all } t \in q^{\mathbb{N}_0}. \quad (2)$$
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Periodicity in this definition is based on the equality of areas lying below the graph of the function at each period.
Definition 2.3 (Adivar, [1])

Let $P \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is said to be $P$-periodic if

$$f \left( q^P t \right) = f(t) \text{ for all } t \in q^{\mathbb{N}_0}. $$
Definition 2.3 (Adivar, [1])

Let $P \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ is said to be $P$-periodic if

$$f(q^P t) = f(t) \text{ for all } t \in q^{\mathbb{N}_0}.$$ 

This definition regards a periodic function to be the one repeating its values after a certain number of steps on $q^{\mathbb{N}_0}$.
For example, the function

\[ h(t) = (-1)^{\frac{\ln t}{\ln q}} \]

on \( q^\mathbb{N}_0 \) is a 2-periodic function according to Definition 2.3, since \( h(q^2t) = h(t) \) holds.
For example, the function

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on \( q^\mathbb{N}_0 \) is a 2-periodic function according to Definition 2.3, since \( h(q^2 t) = h(t) \) holds. On the other hand, the function

\[ g(t) = 1/t \]

is 1-periodic with respect to Definition 2.2 since it satisfies \( qg(qt) = g(t) \).
Theorem 2.4

Let \( f : q^{N_0} \to \mathbb{R} \). Then \( f \) is periodic with respect to Definition 2.2 if and only if \( \tilde{f}(t) = tf(t) \) is periodic with respect to Definition 2.3 with the same period.
Theorem 2.5

The function $x$ is a $P$-periodic solution of the following first order $q$-difference equation

$$D_q x(t) + a(t)x^{\sigma}(t) = f(t, tx(t)), \ t \in q^{N_0}, \quad (3)$$

with respect to Definition 2.2 if and only if $\tilde{x}(t) := tx(t)$ is a $P$-periodic solution of the first order $q$-difference equation

$$D_q \tilde{x}(t) + \tilde{a}(t)\tilde{x}^{\sigma}(t) = \tilde{f}(t, \tilde{x}(t)), \ t \in q^{N_0}, \quad (4)$$

where

$$\tilde{a}(t) := \frac{ta(t) - 1}{qt},$$

and $\tilde{f}(t, \tilde{x}(t)) = tf(t, tx(t))$, with respect to Definition 2.3.
Proof:

Let $x$ be a solution of (3). Then

$$tD_q x(t) + x^\sigma(t) - x^\sigma(t) + ta(t)x^\sigma(t) = tf(t, tx(t)),$$

which implies $\tilde{x}$ solves (4).

The proof that $\tilde{x}$ solves (4) implies $x$ solves (3) is similar. Theorem 2.4 implies that $x$ is $P$-periodic with respect to Definition 2.2 if and only if $\tilde{x}$ is $P$-periodic with respect to Definition 2.3.
Proof:

Let \( x \) be a solution of (3). Then

\[
tD_qx(t) + x^\sigma(t) - x^\sigma(t) + ta(t)x^\sigma(t) = tf(t, tx(t)),
\]

which implies

\[
D_q(tx(t)) + \frac{ta(t) - 1}{qt} qtx^\sigma(t) = tf(t, tx(t)).
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Let $x$ be a solution of (3). Then

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This implies $\tilde{x}$ solves (4).
Proof:

Let \( x \) be a solution of (3). Then

\[
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which implies

\[
D_q(tx(t)) + \frac{ta(t) - 1}{qt} qtx^\sigma(t) = tf(t, tx(t)).
\]

This implies \( \tilde{x} \) solves (4). The proof that \( \tilde{x} \) solves (4) implies \( x \) solves (3) is similar. Theorem 2.4 implies that \( x \) is \( P \)-periodic with respect to Definition 2.2 if and only if \( \tilde{x} \) is \( P \)-periodic with respect to Definition 2.3.
Suppose that \( a : q^\mathbb{N}_0 \to \mathbb{R} \) is a function with \((1 + (q - 1)t a(t)) \neq 0\) for all \( t \in q^\mathbb{N}_0 \). Based on the function \( a \), we define the natural exponential functions

\[
e_a(q^n, q^m) := \prod_{k=m}^{n-1} (1 + (q - 1)q^k a(q^k)) \quad \text{and} \quad e_{\ominus a}(q^n, q^m) := e_a(q^n, q^m)^{-1}.
\]
Suppose that $a : q^{N_0} \to \mathbb{R}$ is a function with $(1 + (q - 1)ta(t)) \neq 0$ for all $t \in q^{N_0}$. Based on the function $a$, we define the natural exponential functions

$$e_a(q^n, q^m) := \prod_{k=m}^{n-1} (1 + (q - 1)q^k a(q^k))$$

and

$$e_{\Theta a}(q^n, q^m) := e_a(q^n, q^m)^{-1}.$$ 

Multiplying both sides of the equations (3) and (4) with $e_a(t, 1)$ and $e_{\tilde{a}}(t, 1)$, respectively, we obtain the following integral equations

$$x(t) = q^P e_{\Theta a} \left(q^P t, t\right) x(t) + q^P \int_t^{q^P t} e_{\Theta a} \left(q^P t, s\right) f \left(s, sx(s)\right) dq s, \quad (5)$$

and

$$\tilde{x}(t) = e_{\Theta \tilde{a}} \left(q^P t, t\right) \tilde{x}(t) + \int_t^{q^P t} e_{\Theta \tilde{a}} \left(q^P t, s\right) f \left(s, \tilde{x}(s)\right) dq s, \quad (6)$$

for $t \in q^{N_0}$. 

Next, the generalizations of (5) and (6) have the form of $q$-Volterra integral equations as follows:

$$x(t) = g(t, tx(t)) + \int_{t}^{q^P t} C(t, s) f(s, sx(s)) \, dq s, \quad (7)$$

and

$$\tilde{x}(t) = \tilde{g}(t, \tilde{x}(t)) + \int_{t}^{q^P t} \tilde{C}(t, s) \tilde{f}(s, \tilde{x}(s)) \, dq s, \quad (8)$$
Next, the generalizations of (5) and (6) have the form of $q$-Volterra integral equations as follows:

$$x(t) = g(t, tx(t)) + \int_{t}^{qP \cdot t} C(t, s) f(s, sx(s)) \, dq s, \quad (7)$$

and

$$\tilde{x}(t) = \tilde{g}(t, \tilde{x}(t)) + \int_{t}^{qP \cdot t} \tilde{C}(t, s) \tilde{f}(s, \tilde{x}(s)) \, dq s, \quad (8)$$

where $g, \tilde{g}, f, \tilde{f} : q^{N_0} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous in their second variable and $C, \tilde{C} : q^{N_0} \times q^{N_0} \rightarrow \mathbb{R}$,

$$\tilde{C}(t, s) = \frac{t}{s} C(t, s), \quad (9)$$

$$\tilde{f}(t, \tilde{x}(t)) = tf(t, tx(t)), \quad (10)$$

and

$$\tilde{g}(t, \tilde{x}(t)) = tg(t, tx(t)). \quad (11)$$
Theorem 2.6

Assume that $C, f, g$ and $x$ satisfy

\[ C\left(q^P t, q^P s\right) = C(t, s), \quad (12) \]
\[ q^P f \left(q^P t, q^P tx \left(q^P t\right)\right) = f \left(t, tx \left(t\right)\right), \quad (13) \]
\[ q^P g \left(q^P t, q^P tx \left(q^P t\right)\right) = g \left(t, tx \left(t\right)\right). \quad (14) \]
Theorem 2.6

Assume that \( C, f, g \) and \( x \) satisfy

\[
C \left( q^P t, q^P s \right) = C(t, s), \quad (12)
\]
\[
q^P f \left( q^P t, q^P t x \left( q^P t \right) \right) = f \left( t, t x \left( t \right) \right), \quad (13)
\]
\[
q^P g \left( q^P t, q^P t x \left( q^P t \right) \right) = g \left( t, t x \left( t \right) \right). \quad (14)
\]

Then \( x(t) \) is a periodic solution of (7) with respect to Definition 2.2 if and only if \( \tilde{x}(t) = t x \left( t \right) \) is a periodic solution of (8) with respect to Definition 2.3.
Proof: Assume (12)-(14) hold and suppose that $x(t)$ solves (7) and is $P$-periodic with respect to Definition 2.2. Let us multiply both sides of (7) by $t$, i.e.,

$$tx(t) = tg(t, tx(t)) + \int_{t}^{qP} tC(t, s) f(s, sx(s)) d_q s,$$

or

$$tx(t) = tg(t, tx(t)) + \int_{t}^{qP} \frac{t}{s} C(t, s) sf(s, sx(s)) d_q s.$$

By employing (9)-(11), we get

$$\ddot{x}(t) = \ddot{g}(t, \ddot{x}(t)) + \int_{t}^{qP} \tilde{C}(t, s) \tilde{f}(s, \ddot{x}(s)) d_q s. \quad (15)$$
Notice that \( \tilde{x}(t) \) is a \( P \)-periodic solution of (15) with respect to Definition 2.3. To show this, consider

\[
\tilde{x}(q^P t) = q^P t \tilde{x}(q^P t) = \tilde{g} (q^P t, \tilde{x}(q^P t)) + \int_{q^P t}^{q^{2P} t} \tilde{C} (q^P t, t) \tilde{f} (s, \tilde{x}(s)) d_q s
\]

\[
= \tilde{g} (q^P t, \tilde{x}(q^P t)) + \int_{q^P t}^{q^P t} \tilde{C} (q^P t, q^P s) \tilde{f} (q^P s, \tilde{x}(q^P s)) d_q s
\]

\[
= q^P t g (q^P t, q^P t x(q^P t)) + \int_{q^P t}^{q^P t} \frac{q^P t}{q^P s} \tilde{C} (q^P t, q^P s) q^P s f (q^P s, q^P s x(q^P s)) d_q s.
\]
Using (12)-(14) we get

\[ \tilde{x}(q^P t) = tg(t, tx(t)) + \int_t^{q^P t} C(t, s) s f(s, sx(s)) d_q s \]

\[ = \tilde{g}(t, \tilde{x}(t)) + \int_t^{q^P t} \tilde{C}(t, s) \tilde{f}(s, \tilde{x}(s)) d_q s. \]

The proof of the necessity part can be done by following a similar procedure used in the sufficiency part, hence, we omit it.
We study the existence of periodic solutions of the following type $q$-Volterra integral equations

$$\tilde{x}(t) = \tilde{g}(t, \tilde{x}(t)) + \int_{t}^{q^P t} \tilde{C}(t, s) \tilde{f}(s, \tilde{x}(s)) \, d_q s, \quad (16)$$
We study the existence of periodic solutions of the following type $q$-Volterra integral equations

$$\ddot{x}(t) = \tilde{g}(t, \dot{x}(t)) + \int_{t}^{q^P t} \tilde{C}(t, s) \tilde{f}(s, \ddot{x}(s)) \, dq s, \quad (16)$$

Let $P$ be the set of all functions defined on $q^{N_0}$ which are $P$-periodic. Then $(P, \|\cdot\|)$ is a Banach space endowed with the norm

$$\|x\| = \max_{t \in [1, q^P]_{q^{N_0}}} |x(t)|$$

where $[1, q^P]_{q^{N_0}} := [1, q^P] \cap q^{N_0}$. 


We have the following assumptions:
We have the following assumptions:

\( \mathcal{K}_1 \) \( \tilde{g} \) satisfies

\[
\tilde{g} \left( q^P t, \tilde{x} \right) = \tilde{g} \left( t, \tilde{x} \right),
\]

for all \( t \in q^{N_0} \);
We have the following assumptions:

\( K_1 \) \( \tilde{g} \) satisfies

\[
\tilde{g} (q^P t, \tilde{x}) = \tilde{g} (t, \tilde{x}),
\]

for all \( t \in q^N_0 \);

\( K_2 \) \( \tilde{C} \) satisfies

\[
\tilde{C} (q^P t, q^P s) = \tilde{C} (t, s),
\]

for all \( (t, s) \in q^N_0 \times q^N_0 \);
We have the following assumptions:

\(\mathcal{K}_1\) \(\tilde{g}\) satisfies

\[ \tilde{g} (q^P t, \tilde{x}) = \tilde{g} (t, \tilde{x}), \]

for all \(t \in q^N_0\);

\(\mathcal{K}_2\) \(\tilde{C}\) satisfies

\[ \tilde{C} (q^P t, q^P s) = \tilde{C} (t, s), \]

for all \((t, s) \in q^N_0 \times q^N_0\);

\(\mathcal{K}_3\) \(\tilde{f}\) satisfies

\[ q^P \tilde{f} (q^P t, \tilde{x}) = \tilde{f} (t, \tilde{x}), \]

for all \(t \in q^N_0\);
We have the following assumptions:

\textbf{K1} \; \tilde{g} \text{ satisfies}

\[\tilde{g} \left( q^P t, \tilde{x} \right) = \tilde{g} \left( t, \tilde{x} \right),\]

for all \( t \in q^{N_0}; \)

\textbf{K2} \; \tilde{C} \text{ satisfies}

\[\tilde{C} \left( q^P t, q^P s \right) = \tilde{C} \left( t, s \right),\]

for all \((t, s) \in q^{N_0} \times q^{N_0};\)

\textbf{K3} \; \tilde{f} \text{ satisfies}

\[q^P \tilde{f} \left( q^P t, \tilde{x} \right) = \tilde{f} \left( t, \tilde{x} \right),\]

for all \( t \in q^{N_0}; \) and

\textbf{K4} \; |\tilde{g} \left( t, \tilde{x} \right) - \tilde{g} \left( t, \tilde{y} \right)| \leq a_1 |\tilde{x} - \tilde{y}|, \; a_1 \in (0, 1).
Lemma 3.1

Assume (K1-K3) and for \( \tilde{\varphi} \in \mathbb{P} \) define the operator \( Q \) as

\[
(Q \tilde{\varphi})(t) := \tilde{g}(t, \tilde{\varphi}(t)) + \int_{t}^{q^N t} \tilde{C}(t, s) \tilde{f}(s, \tilde{\varphi}(s)) \, dq s.
\]  

Then \( Q : \mathbb{P} \rightarrow \mathbb{P} \).
Theorem 3.2 (Krasnosel’skii, [4])

Let $\mathcal{M}$ be a closed convex nonempty subset of a Banach space $(\mathcal{B}, \|\cdot\|)$. 
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Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \| \cdot \|)$. Suppose that $A$ and $B$ map $\mathbb{M}$ into $\mathbb{B}$ such that

(i) $x, y \in \mathbb{M}$ implies $Ax + By \in \mathbb{M}$,
Theorem 3.2 (Krasnosel’skii, [4])

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Theorem 3.2 (Krasnosel’skii, [4])

Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \| \cdot \|)$. Suppose that $A$ and $B$ map $\mathbb{M}$ into $\mathbb{B}$ such that

(i) $x, y \in \mathbb{M}$ implies $Ax + By \in \mathbb{M}$,

(ii) $A$ is a contraction mapping, and

(iii) $B$ is a compact and continuous mapping.
Theorem 3.2 (Krasnosel’skii, [4])

Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that $A$ and $B$ map $\mathbb{M}$ into $\mathbb{B}$ such that

(i) $x, y \in \mathbb{M}$ implies $Ax + By \in \mathbb{M}$,
(ii) $A$ is a contraction mapping, and
(iii) $B$ is a compact and continuous mapping.

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$. 
Now, the operator $Q$ given in (17) can be written as

$$(Q\tilde{\varphi})(t) := (A\tilde{\varphi})(t) + (B\tilde{\varphi})(t),$$

where

$$(A\tilde{\varphi})(t) := \tilde{g}(t, \tilde{\varphi}(t)),$$

and

$$(B\tilde{\varphi})(t) := qt \int_{t}^{\infty} \tilde{C}(t, s) \tilde{f}(s, \tilde{\varphi}(s)) ds.$$
Now, the operator $Q$ given in (17) can be written as

$$(Q\tilde{\varphi})(t) := (A\tilde{\varphi})(t) + (B\tilde{\varphi})(t),$$

where

$$(A\tilde{\varphi})(t) := \tilde{g}(t, \tilde{\varphi}(t)), \quad (18)$$
Now, the operator $Q$ given in (17) can be written as

$$(Q\tilde{\varphi})(t) := (A\tilde{\varphi})(t) + (B\tilde{\varphi})(t),$$

where

$$(A\tilde{\varphi})(t) := \tilde{g}(t, \tilde{\varphi}(t)),$$  \hspace{1cm} (18)

and

$$(B\tilde{\varphi})(t) := \int_{t}^{q^{p}t} \tilde{C}(t, s) \tilde{f}(s, \tilde{\varphi}(s)) \, dq \, s.$$  \hspace{1cm} (19)
Lemma 3.3

Suppose (K4) holds. Then $A : \mathbb{P} \rightarrow \mathbb{P}$ is a contraction mapping.
Lemma 3.3

Suppose \((K4)\) holds. Then \(A : \mathbb{P} \rightarrow \mathbb{P}\) is a contraction mapping.

Lemma 3.4

Assume \((K1-K3)\) hold. Then \(B : \mathbb{P} \rightarrow \mathbb{P}\) is a continuous compact mapping.
Define

\[ \tilde{C} := \max_{(t,s) \in [1,q^P]_q \times [t,q^P t]_q} |\tilde{C}(t, s)|. \]  

(20)
Define

\[ \tilde{C} := \max_{(t,s) \in [1,q^P]_{qN_0} \times [t,q^P t]_{qN_0}} \left| \tilde{C}(t,s) \right|. \quad (20) \]

Define the function \( \bar{F} : \mathbb{R} \to \mathbb{R} \) by

\[ \bar{F}(m) = \sup_{(t,x) \in [1,q^P]_{qN_0} \times [-m,m]} \left| \tilde{f}(t,x) \right|. \quad (21) \]
Theorem 3.5

Assume ($K_1-K_4$). If there exists a positive constant $M_0$ such that

$$\frac{\alpha + \bar{C}\bar{F}(M_0)q^P(q^P-1)}{1-a_1} \leq M_0,$$

where $\alpha = \|g(t,0)\|$, then equation (16) has a $P$-periodic solution in $\Pi_{M_0} := \{\varphi \in \mathbb{P} : \|\varphi\| \leq M_0\}$ with respect to Definition 2.3.
Proof:

For $\tilde{\phi}, \tilde{\psi} \in \Pi_{M_0}$, we have, for $t \in [1, q^P]_{q^N_0}$,

$$|A\tilde{\phi} + B\tilde{\psi}|(t) \leq |\tilde{g}(t, \tilde{\phi}(t))| + \left| \int_t^{q^P t} \tilde{C}(t, s) \tilde{f}(s, \tilde{\phi}(s)) d_q s \right|. \quad (23)$$
Proof:

For $\tilde{\varphi}, \tilde{\psi} \in \Pi_{M_0}$, we have, for $t \in [1, q^P]_{q^{N_0}}$,

$$|A\tilde{\varphi} + B\tilde{\psi}(t)| \leq |\tilde{g}(t, \tilde{\varphi}(t))| + \left| \int_t^{q^P t} \tilde{C}(t, s) \tilde{f}(s, \tilde{\phi}(s)) d_q s \right|. \quad (23)$$

Notice that for $t \in [1, q^P]_{q^{N_0}}$,

$$|\tilde{g}(t, \tilde{\varphi}(t))| \leq |\tilde{g}(t, \tilde{\varphi}(t)) - \tilde{g}(t, 0)| + |g(t, 0)|$$

$$\leq a_1 |\tilde{\varphi}(t)| + \alpha$$

$$\leq a_1 \|\tilde{\varphi}\| + \alpha$$

$$\leq a_1 M_0 + \alpha. \quad (24)$$
Therefore by (22), (23), and (24), we obtain, for \( t \in [1, q^P]_{q^N_0} \),

\[
|A\tilde{\varphi} + B\tilde{\psi}(t) \leq a_1 M_0 + \alpha + \bar{C}\bar{F}(M_0)q^P(q^P - 1) \leq M_0.
\]
Therefore by (22), (23), and (24), we obtain, for $t \in [1, q^P]_{q^{N_0}}$,

$$|A\tilde{\phi} + B\tilde{\psi}|(t) \leq a_1 M_0 + \alpha + \bar{c} \bar{F}(M_0) q^P (q^P - 1) \leq M_0.$$ 

Therefore, for $\tilde{\phi}, \tilde{\psi} \in \Pi_{M_0}$, $\|A\tilde{\phi} + B\tilde{\psi}\| \leq M_0$. So $A\tilde{\phi} + B\tilde{\psi} \in \Pi_{M_0}$, which proves condition (i) of Theorem 3.2.
Therefore by (22), (23), and (24), we obtain, for \( t \in [1, q^P]_{q^N_0} \),

\[
|A\tilde{\varphi} + B\tilde{\psi}(t)| \leq a_1 M_0 + \alpha + \bar{C} \bar{F}(M_0) q^P (q^P - 1) \leq M_0.
\]

Therefore, for \( \tilde{\varphi}, \tilde{\psi} \in \Pi_{M_0} \), \( \|A\tilde{\varphi} + B\tilde{\psi}\| \leq M_0 \). So \( A\tilde{\varphi} + B\tilde{\psi} \in \Pi_{M_0} \), which proves condition (i) of Theorem 3.2. Notice Lemma 3.3 and Lemma 3.4 prove conditions (ii) and (iii) of Theorem 3.2.
Therefore by (22), (23), and (24), we obtain, for \( t \in [1, q^P]_{q^N_0} \),

\[
|A\tilde{\phi} + B\tilde{\psi}|(t) \leq a_1 M_0 + \alpha + \bar{C} \bar{F}(M_0) q^P (q^P - 1) \leq M_0.
\]

Therefore, for \( \tilde{\phi}, \tilde{\psi} \in \Pi_{M_0} \), \( \|A\tilde{\phi} + B\tilde{\psi}\| \leq M_0 \). So \( A\tilde{\phi} + B\tilde{\psi} \in \Pi_{M_0} \), which proves condition (i) of Theorem 3.2. Notice Lemma 3.3 and Lemma 3.4 prove conditions (ii) and (iii) of Theorem 3.2. Therefore there exists a \( P \)-periodic solution of (16) with respect to Definition 2.3.
Example 4.1

Consider the following equation

\[ \tilde{x}(t) = \frac{1}{2} \sin \left( \frac{\ln t}{\ln 2} \pi \right) \tilde{x}(t) + \frac{1}{48e^4} \int_0^{4t} \exp \left( \frac{s}{t} \right) \tilde{\Lambda}(s) \tilde{x}(s) d_2 s, \quad t \in 2^N_0. \]  

(25)

Here

\[ \tilde{\Lambda}(t) = \begin{cases} 
\frac{1}{t}, & \text{if } \log_2 t \text{ is odd} \\
\frac{2}{t}, & \text{if } \log_2 t \text{ is even}. 
\end{cases} \]
\begin{align*}
\check{x}(t) &= \frac{1}{2} \sin \left( \frac{\ln t}{\ln 2} \pi \right) \check{x}(t) + \frac{1}{48e^4} \int_{t}^{4t} \exp \left( \frac{s}{t} \right) \check{\Lambda}(s) \check{x}(s) ds
\end{align*}

Here

\begin{align*}
\check{g}(t, \check{x}(t)) &= \frac{1}{2} \sin \left( \frac{\ln t}{\ln 2} \pi \right) \check{x}(t), \\
\check{C}(t, s) &= \exp \left( \frac{s}{t} \right),
\end{align*}

and

\begin{align*}
\check{f}(t, \check{x}(t)) &= \frac{1}{48e^4} \check{\Lambda}(t) \check{x}(t),
\end{align*}

Observe that assumptions (K1 – K3) are satisfied, and the function \( \check{g} \) satisfies the Lipschitz condition (K4) with constant \( a_1 = 1/2 \). We obtain \( \check{C} = e^4 \) and \( \check{F}(M_0) = (1/e^424)M_0 \), respectively. Also, \( \alpha = 0 \). Then, the inequality (22) is satisfied for any positive constant \( M_0 \). By Theorem 3.5, we conclude that the equation (25) has a 2-periodic solution with respect to Definition 2.3.
Example 4.2

Since (25) has a 2-periodic solution with respect to Definition 2.3, then the integral equation

\[ x(t) = \frac{1}{2} \sin \left( \frac{\ln t}{\ln 2} \pi \right) x(t) + \frac{1}{48e^4} \int_{t}^{4t} \frac{1}{s} \exp \left( \frac{s}{t} \right) \tilde{\Lambda}(s) sx(s) d_2 s, \]

(26)

has a 2-periodic solution with respect to Definition 2.2.
THANK YOU!
Murat Adivar.  
A new periodicity concept for time scales.  

Martin Bohner and Rotchana Chieochan.  
Floquet theory for $q$-difference equations.  

Eric R. Kaufmann and Youssef N. Raffoul.  
Periodic solutions for a neutral nonlinear dynamical equation on a time scale.  

D. R. Smart.  
*Fixed point theorems*.  
Cambridge Tracts in Mathematics, No. 66.