Periodicity in Quantum Calculus

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For $q > 1$, the time scale

$$q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\} = \{1, q, q^2, q^3, q^4, \ldots \},$$

is called the quantum time scale.
Definition 1.2

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, define the forward jump operator

$$\sigma(t) = \inf \left\{ s \in \mathbb{T} : s > t \right\},$$

and the backward jump operator

$$\rho(t) = \sup \left\{ s \in \mathbb{T} : s < t \right\}.$$
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• On $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$, $\rho(t) = t$ and $\mu(t) \equiv 0$.
• On $T = q^{\mathbb{N}_0}$, $\sigma(t) = qt$, $\rho(t) = \frac{t}{q}$ ($t > 1$), and $\mu(t) = (q - 1)t$. 
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- $t$ is left-dense and right-scattered.
- $x$ is isolated.
- $y$ is left-scattered and right-dense.
- $z$ is dense.
Definition 1.4

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T} \setminus \text{sup} \mathbb{T}$. Then if $f$ is continuous at $t$ and $t$ is right-scattered ($t < \sigma(t)$), then $f$ is differentiable at $t$ with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$
**Definition 1.4**

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\[
f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.
\]

Notice this is the slope of the secant line connecting \( (t, f(t)) \) and \( (\sigma(t), f(\sigma(t))) \).
• On $\mathbb{T} = \mathbb{N}$,

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• On $\mathbb{T} = q^{\mathbb{N}_0}$,

$$f^\Delta(t) = D_q f(t) = \frac{f(qt) - f(t)}{(q - 1)t}.$$
Definition 1.5

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T} \setminus \text{sup} \mathbb{T}$. Then if $f$ is continuous at $t$ and $t$ is right-dense ($t = \sigma(t)$), then

$$f^\Delta(t) = \lim_{s \to t} f(t) - f(s),$$

provided the limit exists. Notice the similarity between $f^\Delta$ and $f'$.
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Notice the similarity between $f^\Delta$ and $f'$.

- On $\mathbb{T} = \mathbb{R}$,

$$f^\Delta(t) = f'(t)$$
Theorem 1.6

Assume \( f, g : \mathbb{T} \to \mathbb{R} \) are differentiable at \( t \). Then

\[
(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t),
\]
Theorem 1.6

Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t$. Then

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\]

\[
(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t),
\]

Here \( f^\sigma(t) = f(\sigma(t)) \).
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and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$
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Here $f^\sigma(t) = f(\sigma(t))$. 
On $\mathbb{T} = \mathbb{R}$, the function $y(t) = e^{at}$ is uniquely determined as the solution of the initial value problem

$$y' = ay, \quad y(0) = 1.$$
On $\mathbb{T} = \mathbb{R}$, the function $y(t) = e^{at}$ is uniquely determined as the solution of the initial value problem

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Similarly, on a general time scale $\mathbb{T}$, we define $e_a(t, t_0)$ as the unique solution of the initial value problem

$$y^\Delta = ay, \quad y(t_0) = 1.$$
Let’s find $e_1(t, 0)$ on $T = \mathbb{Z}$. 
Let’s find $e_1(t,0)$ on $\mathbb{T} = \mathbb{Z}$. Try a guess of $e_1(t,0) = a^t$ for some $a$. 

Then $e_1(t,0) = a^t + 1 - a^t = a^t(a - 1) = a^t$ iff $a - 1 = 1$, or $a = 2$. So $e_1(t,0) = 2t$ on $\mathbb{T} = \mathbb{Z}$. 

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Let’s find $e_1(t,0)$ on $\mathbb{T} = \mathbb{Z}$. Try a guess of $e_1(t,0) = a^t$ for some $a$. Then

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$$e_1^\Delta(t,0) = a^{t+1} - a^t = a^t(a - 1) = a^t = e_1(t,0)$$

iff $a - 1 = 1$, or $a = 2$. So $e_1(t,0) = 2^t$ on $\mathbb{T} = \mathbb{N}$. 
• On $2^{\mathbb{N}_0}$,

$$e_1(t, 1) = \prod_{s \in [1, t) \cap T} (1 + s) = 2 \cdot 3 \cdot 5 \cdots \left(\frac{t}{2} + 1\right).$$
• On $2^\mathbb{N}_0$,

\[ e_1(t, 1) = \prod_{s \in [1, t) \cap \mathbb{T}} (1 + s) = 2 \cdot 3 \cdot 5 \cdots \left( \frac{t}{2} + 1 \right). \]

So, for example,

\[ e_1(2, 1) = 2, \quad e_1(4, 1) = 2 \cdot 3 = 6, \quad e_1(8, 1) = 2 \cdot 3 \cdot 5 = 30. \]
• On $2^\mathbb{N}_0$,

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Notice

$$e_1^A(2, 1) = \frac{6 - 2}{2} = 2 = e_1(2, 1),$$
• On $2^\mathbb{N}_0$, 

$$e_1(t, 1) = \prod_{s \in [1, t] \cap \mathbb{T}} (1 + s) = 2 \cdot 3 \cdot 5 \cdots \left(\frac{t}{2} + 1\right).$$

So, for example,

$$e_1(2, 1) = 2, \quad e_1(4, 1) = 2 \cdot 3 = 6, \quad e_1(8, 1) = 2 \cdot 3 \cdot 5 = 30.$$

Notice

$$e_1^\Delta(2, 1) = \frac{6 - 2}{2} = 2 = e_1(2, 1),$$

and

$$e_1^\Delta(4, 1) = \frac{30 - 6}{4} = 6 = e_1(4, 1).$$
Definition 1.7

If $F^\Delta (t) = f(t)$, define the indefinite integral of $f$ by

$$\int f(t) \Delta t = F(t) + C,$$

and the definite integral of $f$ from $a$ to $b$ by

$$\int_a^b f(t) \Delta t = F(b) - F(a).$$
• If $T = \mathbb{R}$,
• If $T = \mathbb{R}$, 

$$
\int_a^b f(t) \Delta t = \int_a^b f(t) dt.
$$
• If $T = \mathbb{R}$,

$$\int_{a}^{b} f(t) \Delta t = \int_{a}^{b} f(t) dt.$$  

• If $T = \mathbb{N}$,
• If $T = \mathbb{R}$,

$$\int_a^b f(t) \Delta t = \int_a^b f(t) dt.$$ 

• If $T = \mathbb{N}$,

$$\int_a^b f(t) \Delta t = \sum_{t=a}^{b-1} f(t).$$
• If $T = q^{N_0}$,
\begin{itemize}
  \item If $T = q^{\mathbb{N}_0}$,

\begin{equation}
\int_{q^m}^{q^n} f(s) \Delta s = \int_{q^m}^{q^n} f(s) dq s = (q - 1) \sum_{k=m}^{n-1} q^k f(q^k). 
\end{equation}
\end{itemize}
Definition 2.1

A time scale, denoted by $\mathbb{T}$, is said to be additively $T$-periodic if there exists a $T > 0$ such that $t \pm T \in \mathbb{T}$ for all $t \in \mathbb{T}$ (see [3]).
Definition 2.1

A time scale, denoted by $\mathbb{T}$, is said to be additively $T$-periodic if there exists a $T > 0$ such that $t \pm T \in \mathbb{T}$ for all $t \in \mathbb{T}$ (see [3]). A function $f$ on additively periodic domain $\mathbb{T}$ with period $T$ is said to be periodic with period $P$ if there exists an $n \in \mathbb{N}$ such that $P = nT$, $f(t \pm P) = f(t)$ for all $t \in \mathbb{T}$, and $P$ is the smallest number such that $f(t \pm P) = f(t)$. 
Notice, since the time scale $q^\mathbb{N}_0$ is not additive, we cannot define periodicity on $q^\mathbb{N}_0$ in a same way we do on additively periodic domains. Recently, two different definitions of periodicity on $q^\mathbb{N}_0$ have arose. We wish to show a relationship between these two definitions of periodicity. We will then show the existence of $P$-periodic solutions of a $q$-Volterra integral equation.
Definition 2.2 (Bohner and Chieochan, [2])

Let $P \in \mathbb{N}$. A function $f : q^\mathbb{N}_0 \rightarrow \mathbb{R}$ is said to be $P$-periodic if

$$f(t) = q^P f(q^P t) \quad \text{for all } t \in q^\mathbb{N}_0.$$  \hspace{1cm} (2)
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(2)

Periodicity in this definition is based on the equality of areas lying below the graph of the function at each period.
Definition 2.3 (Adivar, [1])

Let $P \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ is said to be $P$-periodic if

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Definition 2.3 (Adivar, [1])

Let $P \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ is said to be $P$-periodic if

$$f \left( q^P t \right) = f(t) \text{ for all } t \in q^{\mathbb{N}_0}.$$  

This definition regards a periodic function to be the one repeating its values after a certain number of steps on $q^{\mathbb{N}_0}$. 
For example, the function

\[ h(t) = (-1)^{\frac{\ln t}{\ln q}} \]

on \( q^{\mathbb{N}_0} \) is a 2-periodic function according to Definition 2.3, since \( h(q^2 t) = h(t) \) holds.
For example, the function

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on \( q^{\mathbb{N}_0} \) is a 2-periodic function according to Definition 2.3, since \( h(q^2 t) = h(t) \) holds. On the other hand, the function

\[ g(t) = 1/t \]

is 1-periodic with respect to Definition 2.2 since it satisfies \( qg(qt) = g(t) \).
Theorem 2.4

Let \( f : q^{N_0} \rightarrow \mathbb{R} \). Then \( f \) is periodic with respect to Definition 2.2 if and only if \( \tilde{f}(t) = tf(t) \) is periodic with respect to Definition 2.3 with the same period.
Theorem 2.5

The function $x$ is a $P$-periodic solution of the following first order $q$-difference equation

$$D_q x(t) + a(t)x^\sigma(t) = f(t, tx(t)), \quad t \in q^{N_0},$$  \hspace{1cm} (3)

with respect to Definition 2.2 if and only if $\tilde{x}(t) := tx(t)$ is a $P$-periodic solution of the first order $q$-difference equation

$$D_q \tilde{x}(t) + \tilde{a}(t)\tilde{x}^\sigma(t) = \tilde{f}(t, \tilde{x}(t)), \quad t \in q^{N_0},$$  \hspace{1cm} (4)

where

$$\tilde{a}(t) := \frac{ta(t) - 1}{qt},$$

and $\tilde{f}(t, \tilde{x}(t)) = tf(t, tx(t))$, with respect to Definition 2.3.
Proof:

Let $x$ be a solution of (3). Then

$$tD_q x(t) + x^\sigma(t) - x^\sigma(t) + ta(t)x^\sigma(t) = tf(t, tx(t)),$$

which implies $\tilde{x}$ solves (4).

The proof that $\tilde{x}$ solves (4) implies $x$ solves (3) is similar. Theorem 2.4 implies that $x$ is $P$-periodic with respect to Definition 2.2 if and only if $\tilde{x}$ is $P$-periodic with respect to Definition 2.3.
Proof:

Let $x$ be a solution of (3). Then

$$tD_q x(t) + x^\sigma(t) - x^\sigma(t) + ta(t)x^\sigma(t) = tf(t, tx(t)),$$

which implies

$$D_q(tx(t)) + \frac{ta(t) - 1}{qt} qtx^\sigma(t) = tf(t, tx(t)).$$
Proof:

Let $x$ be a solution of (3). Then

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This implies $\tilde{x}$ solves (4).
Proof:

Let \( x \) be a solution of (3). Then

\[
 t D_q x(t) + x^\sigma(t) - x^\sigma(t) + t a(t) x^\sigma(t) = t f(t, t x(t)),
\]

which implies

\[
 D_q(tx(t)) + \frac{ta(t) - 1}{qt} qtx^\sigma(t) = tf(t, tx(t)).
\]

This implies \( \tilde{x} \) solves (4). The proof that \( \tilde{x} \) solves (4) implies \( x \) solves (3) is similar. Theorem 2.4 implies that \( x \) is \( P \)-periodic with respect to Definition 2.2 if and only if \( \tilde{x} \) is \( P \)-periodic with respect to Definition 2.3.
Suppose that $a: \mathbb{Q}^N_0 \to \mathbb{R}$ is a function with $(1 + (q - 1)ta(t)) \neq 0$ for all $t \in \mathbb{Q}^N_0$. Based on the function $a$, we define the natural exponential functions

$$e_a(q^n, q^m) := \prod_{k=m}^{n-1} (1 + (q - 1)q^k a(q^k))$$

and

$$\Theta_a(q^n, q^m) := e_a(q^n, q^m)^{-1}.$$
Suppose that $a : q^{N_0} \rightarrow \mathbb{R}$ is a function with $(1 + (q - 1)ta(t)) \neq 0$ for all $t \in q^{N_0}$. Based on the function $a$, we define the natural exponential functions

$$e_a(q^n, q^m) := \prod_{k=m}^{n-1} (1 + (q - 1)q^k a(q^k))$$

and

$$e_{\Theta a}(q^n, q^m) := e_a(q^n, q^m)^{-1}.$$ 

Multiplying both sides of the equations (3) and (4) with $e_a(t, 1)$ and $e_{\Theta a}(t, 1)$, respectively, we obtain the following integral equations

$$x(t) = q^P e_{\Theta a} \left(q^P t, t\right) x(t) + q^P \int_t^{q^P t} e_{\Theta a} \left(q^P t, s\right) f(s, sx(s)) d_q s, \quad (5)$$

and

$$\ddot{x}(t) = e_{\Theta \ddot{a}} \left(q^P t, t\right) \ddot{x}(t) + \int_t^{q^P t} e_{\Theta \ddot{a}} \left(q^P t, s\right) f(s, \ddot{x}(s)) d_q s, \quad (6)$$

for $t \in q^{N_0}$. 
Next, the generalizations of (5) and (6) have the form of $q$-Volterra integral equations as follows:

$$\begin{align*}
x(t) &= g(t, tx(t)) + \int_t^{qt} C(t, s) f(s, sx(s)) \, dq s, \\
\tilde{x}(t) &= \tilde{g}(t, \tilde{x}(t)) + \int_t^{qt} \tilde{C}(t, s) \tilde{f}(s, \tilde{x}(s)) \, dq s,
\end{align*}$$

(7)

and

(8)
Next, the generalizations of (5) and (6) have the form of $q$-Volterra integral equations as follows:

\[ x(t) = g(t, tx(t)) + \int_{t}^{qP t} C(t, s) f(s, sx(s)) \, dq \, s, \quad (7) \]

and

\[ \tilde{x}(t) = \tilde{g}(t, \tilde{x}(t)) + \int_{t}^{qP t} \tilde{C}(t, s) \tilde{f}(s, \tilde{x}(s)) \, dq \, s, \quad (8) \]

where $g, \tilde{g}, f, \tilde{f} : q^{N_0} \times \mathbb{R} \to \mathbb{R}$ are continuous in their second variable and $C, \tilde{C} : q^{N_0} \times q^{N_0} \to \mathbb{R}$,

\[ \tilde{C}(t, s) = \frac{t}{s} C(t, s), \quad (9) \]

\[ \tilde{f}(t, \tilde{x}(t)) = tf(t, tx(t)), \quad (10) \]

and

\[ \tilde{g}(t, \tilde{x}(t)) = tg(t, tx(t)). \quad (11) \]
Theorem 2.6

Assume that $C$, $f$, $g$ and $x$ satisfy

$$C \left( q^P t, q^P s \right) = C(t, s), \quad (12)$$

$$q^P f \left( q^P t, q^P tx \left( q^P t \right) \right) = f \left( t, tx \left( t \right) \right), \quad (13)$$

$$q^P g \left( q^P t, q^P tx \left( q^P t \right) \right) = g \left( t, tx \left( t \right) \right). \quad (14)$$
Theorem 2.6

Assume that \( C, f, g \) and \( x \) satisfy

\[
C \left( q^P t, q^P s \right) = C(t, s),
\]
\[
q^P f \left( q^P t, q^P tx \left( q^P t \right) \right) = f \left( t, tx(t) \right),
\]
\[
q^P g \left( q^P t, q^P tx \left( q^P t \right) \right) = g \left( t, tx(t) \right).
\]

Then \( x(t) \) is a periodic solution of (7) with respect to Definition 2.2 if and only if \( \tilde{x}(t) = tx(t) \) is a periodic solution of (8) with respect to Definition 2.3.
Proof: Assume (12)-(14) hold and suppose that $x(t)$ solves (7) and is $P$-periodic with respect to Definition 2.2. Let us multiply both sides of (7) by $t$, i.e.,

$$tx(t) = tg(t, tx(t)) + \int_t^{q^P t} tC(t, s) f(s, sx(s)) \, dq \, s,$$

or

$$tx(t) = tg(t, tx(t)) + \int_t^{q^P t} \frac{t}{s} C(t, s) sf(s, sx(s)) \, dq \, s.$$

By employing (9)-(11), we get

$$\tilde{x}(t) = \tilde{g}(t, \tilde{x}(t)) + \int_t^{q^P t} \tilde{C}(t, s) \tilde{f}(s, \tilde{x}(s)) \, dq \, s. \quad (15)$$
Notice that $\tilde{x}(t)$ is a $P$-periodic solution of (15) with respect to Definition 2.3. To show this, consider

$$\tilde{x}(q^P t) = q^P t x (q^P t) = \tilde{g} (q^P t, \tilde{x} (q^P t)) + \int_{q^P t}^{q^2 P t} \tilde{C} (q^P t, s) \tilde{f} (s, \tilde{x} (s)) \, dq \, s$$

$$= \tilde{g} (q^P t, \tilde{x} (q^P t)) + \int_{q^P t}^{q^P t} \tilde{C} (q^P t, q^P s) \tilde{f} (q^P s, \tilde{x} (q^P s)) \, dq \, s$$

$$= q^P t g (q^P t, q^P t x (q^P t))$$

$$+ \int_{q^P t}^{q^P t} \frac{q^P t}{q^P s} C (q^P t, q^P s) q^P s f (q^P s, q^P s x (q^P s)) \, dq \, s.$$
Using (12)-(14) we get

\[ \tilde{x}(q^p t) = tg(t, tx(t)) + \int_t^{q^p t} \frac{t}{s} C(t, s) f(s, sx(s)) \, dq s \]

\[ = \tilde{g}(t, \tilde{x}(t)) + \int_t^{q^p t} \tilde{C}(t, s) \tilde{f}(s, \tilde{x}(s)) \, dq s. \]

The proof of the necessity part can be done by following a similar procedure used in the sufficiency part, hence, we omit it.
We study the existence of periodic solutions of the following type $q$-Volterra integral equations

$$
\tilde{x}(t) = \tilde{g}(t, \tilde{x}(t)) + \int_{t}^{qP_t} \tilde{C}(t, s) \tilde{f}(s, \tilde{x}(s)) \, dq_s,
$$

(16)
We study the existence of periodic solutions of the following type $q$-Volterra integral equations

$$
\tilde{x}(t) = \tilde{g}(t, \tilde{x}(t)) + \int_{t}^{q^P t} \tilde{C}(t, s) \tilde{f}(s, \tilde{x}(s)) \, dq s,
$$

(16)

Let $\mathbb{P}$ be the set of all functions defined on $q^{\mathbb{N}_0}$ which are $P$-periodic. Then $(\mathbb{P}, \|\cdot\|)$ is a Banach space endowed with the norm

$$
\|x\| = \max_{t \in [1, q^P]_{q^{\mathbb{N}_0}}} |x(t)|
$$

where $[1, q^P]_{q^{\mathbb{N}_0}} := [1, q^P] \cap q^{\mathbb{N}_0}$. 
We have the following assumptions:
We have the following assumptions:

\( K1 \) \( \tilde{g} \) satisfies

\[
\tilde{g} (q^P t, \tilde{x}) = \tilde{g} (t, \tilde{x}),
\]

for all \( t \in q^{N_0} \);
We have the following assumptions:

\[ K_1 \tilde{g} \text{ satisfies } \tilde{g}(q^P t, \tilde{x}) = \tilde{g}(t, \tilde{x}), \]

for all \( t \in q^{N_0}; \)

\[ K_2 \tilde{C} \text{ satisfies } \tilde{C}(q^P t, q^P s) = \tilde{C}(t, s), \]

for all \( (t, s) \in q^{N_0} \times q^{N_0}; \)
We have the following assumptions:

\section*{K1} \( \tilde{g} \) satisfies

\[ \tilde{g} (q^P t, \tilde{x}) = \tilde{g} (t, \tilde{x}), \]

for all \( t \in q^{N_0}; \)

\section*{K2} \( \tilde{C} \) satisfies

\[ \tilde{C} (q^P t, q^P s) = \tilde{C} (t, s), \]

for all \( (t, s) \in q^{N_0} \times q^{N_0}; \)

\section*{K3} \( \tilde{f} \) satisfies

\[ q^P \tilde{f} (q^P t, \tilde{x}) = \tilde{f} (t, \tilde{x}), \]

for all \( t \in q^{N_0}; \)

and

\[ |\tilde{g} (t, \tilde{x}) - \tilde{g} (t, \tilde{y})| \leq a_1 |\tilde{x} - \tilde{y}|, \]

\( a_1 \in (0, 1). \)
We have the following assumptions:

\( \mathcal{K}1 \) \( \tilde{g} \) satisfies

\[
\tilde{g} (q^P t, \tilde{x}) = \tilde{g} (t, \tilde{x}),
\]

for all \( t \in q^{N_0} \);

\( \mathcal{K}2 \) \( \tilde{C} \) satisfies

\[
\tilde{C} (q^P t, q^P s) = \tilde{C} (t, s),
\]

for all \( (t, s) \in q^{N_0} \times q^{N_0} \);

\( \mathcal{K}3 \) \( \tilde{f} \) satisfies

\[
q^P \tilde{f} (q^P t, \tilde{x}) = \tilde{f} (t, \tilde{x}),
\]

for all \( t \in q^{N_0} \); and

\( \mathcal{K}4 \) \[
|\tilde{g} (t, \tilde{x}) - \tilde{g} (t, \tilde{y})| \leq a_1 |\tilde{x} - \tilde{y}|, \ a_1 \in (0, 1).
\]
Lemma 3.1

Assume (K1-K3) and for $\tilde{\phi} \in \mathbb{P}$ define the operator $Q$ as

$$(Q\tilde{\phi})(t) := \tilde{g}(t, \tilde{\phi}(t)) + \int_{t}^{qP t} \tilde{C}(t, s) \tilde{f}(s, \tilde{\phi}(s)) \, dq s. \quad (17)$$

Then $Q : \mathbb{P} \rightarrow \mathbb{P}$. 
Theorem 3.2 (Krasnosel’skii, [4])

Let \( M \) be a closed convex nonempty subset of a Banach space \((\mathbb{B}, \| \cdot \|)\).
Theorem 3.2 (Krasnosel’skii, [4])

Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \| \cdot \|)$. Suppose that $A$ and $B$ map $\mathbb{M}$ into $\mathbb{B}$ such that

(i) $x, y \in \mathbb{M}$ implies $Ax + By \in \mathbb{M}$,
Theorem 3.2 (Krasnosel’skii, [4])

Let $\mathcal{M}$ be a closed convex nonempty subset of a Banach space $(\mathcal{B}, \|\cdot\|)$. Suppose that $A$ and $B$ map $\mathcal{M}$ into $\mathcal{B}$ such that

(i) $x, y \in \mathcal{M}$ implies $Ax + By \in \mathcal{M}$,

(ii) $A$ is a contraction mapping, and
Theorem 3.2 (Krasnosel’skii, [4])

Let $M$ be a closed convex nonempty subset of a Banach space $(B, \|\cdot\|)$. Suppose that $A$ and $B$ map $M$ into $B$ such that

(i) $x, y \in M$ implies $Ax + By \in M$,

(ii) $A$ is a contraction mapping, and

(iii) $B$ is a compact and continuous mapping.
Theorem 3.2 (Krasnosel’skii, [4])

Let \( \mathbb{M} \) be a closed convex nonempty subset of a Banach space \( (\mathbb{B}, \| \cdot \|) \). Suppose that \( A \) and \( B \) map \( \mathbb{M} \) into \( \mathbb{B} \) such that

(i) \( x, y \in \mathbb{M} \) implies \( Ax + By \in \mathbb{M} \),

(ii) \( A \) is a contraction mapping, and

(iii) \( B \) is a compact and continuous mapping.

Then there exists \( z \in \mathbb{M} \) with \( z = Az + Bz \).
Now, the operator $Q$ given in (17) can be written as

$$(Q\tilde{\phi})(t) := (A\tilde{\phi})(t) + (B\tilde{\phi})(t),$$
Now, the operator $Q$ given in (17) can be written as

$$(Q\tilde{\varphi})(t) := (A\tilde{\varphi})(t) + (B\tilde{\varphi})(t),$$

where

$$(A\tilde{\varphi})(t) := \tilde{g}(t, \tilde{\varphi}(t)), \quad (18)$$
Now, the operator $Q$ given in (17) can be written as

$$(Q\tilde{\varphi})(t) := (A\tilde{\varphi})(t) + (B\tilde{\varphi})(t),$$

where

$$(A\tilde{\varphi})(t) := \tilde{g}(t, \tilde{\varphi}(t)), \quad (18)$$

and

$$(B\tilde{\varphi})(t) := \int_t^{q^P t} \tilde{C}(t, s) \tilde{f}(s, \tilde{\varphi}(s)) d_q s. \quad (19)$$
Lemma 3.3

Suppose \((K4)\) holds. Then \(A : \mathbb{P} \to \mathbb{P}\) is a contraction mapping.
Lemma 3.3

Suppose (K4) holds. Then $A : \mathbb{P} \rightarrow \mathbb{P}$ is a contraction mapping.

Lemma 3.4

Assume (K1-K3) hold. Then $B : \mathbb{P} \rightarrow \mathbb{P}$ is a continuous compact mapping.
Define

\[ \tilde{C} := \max_{(t,s) \in [1, q^P]_q \times [t, q^P t]_q} \left| \tilde{C}(t, s) \right|. \]  

(20)
Define

\[ \tilde{C} := \max_{(t,s) \in [1, q^P]_{qN_0} \times [t, q^P t]_{qN_0}} |\tilde{C}(t, s)|. \]  

(20)

Define the function \( \tilde{F} : \mathbb{R} \to \mathbb{R} \) by

\[ \tilde{F}(m) = \sup_{(t,x) \in [1, q^P]_{qN_0} \times [-m,m]} |\tilde{f}(t, x)|. \]  

(21)
Theorem 3.5

Assume (K1-K4). If there exists a positive constant $M_0$ such that

$$\alpha + \bar{C}\bar{F}(M_0)q^p(q^p - 1) \leq M_0,$$

(22)

where $\alpha = \|g(t,0)\|$, then equation (16) has a $P$-periodic solution in $\Pi_{M_0} := \{\varphi \in P : \|\varphi\| \leq M_0\}$ with respect to Definition 2.3.
**Proof:**

For \( \tilde{\varphi}, \tilde{\psi} \in \Pi_{M_0} \), we have, for \( t \in [1, q^P]_{q^{N_0}} \),

\[
|A\tilde{\varphi} + B\tilde{\psi}|(t) \leq |\tilde{g}(t, \tilde{\varphi}(t))| + \left| \int_t^{q^P t} \tilde{C}(t, s)\tilde{f}(s, \tilde{\varphi}(s))d_q s \right|. \quad (23)
\]
Proof:

For $\tilde{\varphi}, \tilde{\psi} \in \Pi_{M_0}$, we have, for $t \in [1, q^P]_{q^N_0}$,

$$|A\tilde{\varphi} + B\tilde{\psi}|(t) \leq |\tilde{g}(t, \tilde{\varphi}(t))| + \left| \int_t^{q^P t} \tilde{C}(t, s)\tilde{f}(s, \tilde{\varphi}(s))d_q s \right|. \quad (23)$$

Notice that for $t \in [1, q^P]_{q^N_0}$,

$$|\tilde{g}(t, \tilde{\varphi}(t))| \leq |\tilde{g}(t, \tilde{\varphi}(t)) - \tilde{g}(t, 0)| + |g(t, 0)|$$

$$\leq a_1 |\tilde{\varphi}(t)| + \alpha$$

$$\leq a_1 \|\tilde{\varphi}\| + \alpha$$

$$\leq a_1 M_0 + \alpha. \quad (24)$$
Therefore by (22), (23), and (24), we obtain, for \( t \in [1, q^p]_{q^N_0} \),

\[
|A\tilde{\phi} + B\tilde{\psi}|(t) \leq a_1 M_0 + \alpha + \bar{C}\bar{F}(M_0)q^P(q^P - 1) \leq M_0.
\]
Therefore by (22), (23), and (24), we obtain, for \( t \in [1, q^P]_{q^N_0} \),

\[
|A\tilde{\phi} + B\tilde{\psi}|(t) \leq a_1 M_0 + \alpha + \bar{C}\bar{F}(M_0)q^P(q^P - 1) \leq M_0.
\]

Therefore, for \( \tilde{\phi}, \tilde{\psi} \in \Pi_{M_0} \), \( \|A\tilde{\phi} + B\tilde{\psi}\| \leq M_0 \). So \( A\tilde{\phi} + B\tilde{\psi} \in \Pi_{M_0} \), which proves condition (i) of Theorem 3.2.
Therefore by (22), (23), and (24), we obtain, for \( t \in [1, q^P]_{q^{N_0}} \),

\[
|A\tilde{\varphi} + B\tilde{\psi}(t)| \leq a_1 M_0 + \alpha + \bar{C}\bar{F}(M_0)q^P(q^P - 1) \leq M_0.
\]

Therefore, for \( \tilde{\varphi}, \tilde{\psi} \in \Pi_{M_0} \), \( \|A\tilde{\varphi} + B\tilde{\psi}\| \leq M_0 \). So \( A\tilde{\varphi} + B\tilde{\psi} \in \Pi_{M_0} \), which proves condition (i) of Theorem 3.2. Notice Lemma 3.3 and Lemma 3.4 prove conditions (ii) and (iii) of Theorem 3.2.
Therefore by (22), (23), and (24), we obtain, for $t \in [1, q^P]_{q^N_0}$,

$$|A\tilde{\phi} + B\tilde{\psi}|(t) \leq a_1 M_0 + \alpha + \bar{C}\bar{F}(M_0)q^P(q^P - 1) \leq M_0.$$ 

Therefore, for $\tilde{\phi}, \tilde{\psi} \in \Pi_{M_0}$, $\|A\tilde{\phi} + B\tilde{\psi}\| \leq M_0$. So $A\tilde{\phi} + B\tilde{\psi} \in \Pi_{M_0}$, which proves condition (i) of Theorem 3.2. Notice Lemma 3.3 and Lemma 3.4 prove conditions (ii) and (iii) of Theorem 3.2. Therefore there exists a $P$-periodic solution of (16) with respect to Definition 2.3.
Example 4.1

Consider the following equation

$$\ddot{x}(t) = \frac{1}{2} \sin \left( \frac{\ln t}{\ln 2} \pi \right) \dot{x}(t) + \frac{1}{48e^4} \int_{t}^{4t} \exp \left( \frac{s}{t} \right) \tilde{\Lambda}(s) \tilde{x}(s) d_2 s, \quad t \in 2^{\mathbb{N}_0}. \quad (25)$$

Here

$$\tilde{\Lambda}(t) = \begin{cases} 
1/t, & \text{if } \log_2 t \text{ is odd} \\
2/t, & \text{if } \log_2 t \text{ is even}
\end{cases}$$
\[ \tilde{x}(t) = \frac{1}{2} \sin \left( \frac{\ln t}{\ln 2} \pi \right) \tilde{x}(t) + \frac{1}{48e^4} \int_0^{4t} \exp \left( \frac{s}{t} \right) \tilde{\Lambda}(s) \tilde{x}(s) \, ds \]

Here

\[ \tilde{g} (t, \tilde{x}(t)) = \frac{1}{2} \sin \left( \frac{\ln t}{\ln 2} \pi \right) \tilde{x}(t), \]

\[ \tilde{C} (t, s) = \exp \left( \frac{s}{t} \right), \]

and

\[ \tilde{f} (t, \tilde{x}(t)) = \frac{1}{48e^4} \tilde{\Lambda}(t) \tilde{x}(t), \]

Observe that assumptions (K1 – K3) are satisfied, and the function \( \tilde{g} \) satisfies the Lipschitz condition (K4) with constant \( a_1 = 1/2 \). We obtain \( \tilde{C} = e^4 \) and \( \tilde{F} (M_0) = (1/e^4 24) M_0 \), respectively. Also, \( \alpha = 0 \). Then, the inequality (22) is satisfied for any positive constant \( M_0 \). By Theorem 3.5, we conclude that the equation (25) has a 2-periodic solution with respect to Definition 2.3.
Example 4.2

Since (25) has a 2-periodic solution with respect to Definition 2.3, then the integral equation

\[ x(t) = \frac{1}{2} \sin \left( \frac{\ln t}{\ln 2} \pi \right) x(t) + \frac{1}{48e^4} \int_t^{4t} \frac{1}{t} \exp \left( \frac{s}{t} \right) \tilde{\Lambda}(s) sx(s) d_2 s, \]

(26)

has a 2-periodic solution with respect to Definition 2.2.
THANK YOU!
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