Geometric Flows

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Curvature of a Plane Curve

Let $C : S^1 \to \mathbb{R}^2$ be a smooth closed curve
Let \( C : S^1 \rightarrow \mathbb{R}^2 \) be a smooth closed curve.

\[
T(s) = \frac{dC}{ds}
\]
Curvature of a Plane Curve

Let $C : S^1 \to \mathbb{R}^2$ be a smooth closed curve

$s$: arclength

$$T(s) = \frac{dC}{ds}$$

$$\frac{dT}{ds} \perp T(s)$$

$$\frac{dT}{ds} = k(s)N(s)$$

$k(s)$: Curvature at $C(s)$
Curvature of a Plane Curve

\[ C(s) = (r \cos \frac{s}{r}, r \sin \frac{s}{r}) \]

\[ T(s) = (-\sin \frac{s}{r}, \cos \frac{s}{r}) \quad N(s) = (-\cos \frac{s}{r}, -\sin \frac{s}{r}) \]

\[ \frac{dT}{ds}(s) = \frac{1}{r}(-\cos \frac{s}{r}, -r \sin \frac{s}{r}) \]

\[ = \frac{1}{r}N(s) \]

\[ k = \frac{1}{r} \]
Curvature of a Plane Curve

\[ N(s) \]

\[ \frac{dT}{ds} \]
Curvature of a Plane Curve

\[ k < 0 \]

\[ N(s) \]

\[ \frac{dT}{ds} \]
Curvature of a Plane Curve

\[ k < 0 \]

\[ \frac{dT}{ds} \]

\[ N(s) \]

\[ k > 0 \]
Curve Shortening Flow
Curve Shortening Flow
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$S^1 \rightarrow \text{Complex Curve}$
Curve Shortening Flow

\[ S^1 \times [0, T) \quad \xrightarrow{C(s,t)} \quad C(s,0) = C(s) \]
Curve Shortening Flow

$S^1 \times [0, T)$

$C(s,t)$

$\frac{\partial C}{\partial t}(s, t) = k(s, t)N(s, t)$
Curve Shortening Flow

\[ \frac{\partial C}{\partial t}(s, t) = k(s, t)N(s, t) \]
Curve Shortening Flow

\[ C(s) : \quad S^1 \longrightarrow \mathbb{R}^2 \]

Curve shortening flow

\[ C(s,t) : \quad S^1 \times [0, T) \longrightarrow \mathbb{R}^2 \]

\[ \frac{\partial C}{\partial t} = kN, \quad C(s,0) = C(s) \]
Curve Shortening Flow

\[
\frac{\partial C}{\partial t} = kN
\]

\(k < 0\)

\(k > 0\)
Curve Shortening Flow

\[ C(s) = (\cos s, \sin s) : S^1 \rightarrow \mathbb{R}^2 \]

\[ C(s, t) = \sqrt{1 - 2t} (\cos s, \sin s) \]

Curvature of \( C(\cdot, t) \) at \( s \): \( k(s, t) = \frac{1}{\sqrt{1-2t}} \)

Inward normal vector of \( C(\cdot, t) \) at \( s \): \( N(s, t) = -(\cos s, \sin s) \)

\[ \frac{\partial C}{\partial t} = kN \]
Curve Shortening Flow

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Curve Shortening Flow

\[ C(s, t) = \sqrt{1 - 2t} (\cos s, \sin s) \quad : \]

\[ S^1 \times [0, \frac{1}{2}) \]

\[ t=0 \]
Curve Shortening Flow

\[ C(s, t) = \sqrt{1 - 2t} (\cos s, \sin s) \]

\[ S^1 \times [0, \frac{1}{2}) \]
Theorem (Short Time Existence and Uniqueness)

Let $C(s) : S^1 \to \mathbb{R}^2$ be smooth. Then there exists $T > 0$ and a unique smooth map $C(s, t) : S^1 \times [0, T) \to \mathbb{R}^2$ such that

$$\frac{\partial C}{\partial t} = kN, \quad C(s, 0) = C(s)$$
\[ C(s, t) = C(s) + r(s, t)N(s, 0) \]
Proof

\[ C(s, t) = C(s) + r(s, t)N(s, 0) \]

\[ \frac{\partial r}{\partial t} = \frac{1 - kr}{(1 - kr)^2 + r^2} \frac{\partial^2 r}{\partial s^2} + \text{lower order terms}, \]

where \( k = k(s, 0). \)
Why shortening?

\[ r : \text{arclength} \quad T : \text{unit tangent vector} \]

\[ L(t) = \text{length of } C(\cdot, t) = \int_{S^1} dr = \int_{S^1} \left< \frac{\partial C}{\partial s}, \frac{\partial C}{\partial s} \right>^{\frac{1}{2}} ds \]

\[ L'(t) = \frac{\partial}{\partial t} \int_{S^1} \left< \frac{\partial^2 C}{\partial t \partial s}, \frac{\partial C}{\partial s} \right>^{\frac{1}{2}} ds \]

\[ = \int_{S^1} \left< \frac{\partial^2 C}{\partial t \partial s}, T(s) \right> ds = \int_{S^1} \left< \frac{\partial^2 C}{\partial s \partial t}, T(s) \right> ds \]

\[ = \int_{S^1} \left< \frac{\partial}{\partial s} (kN), T(s) \right> ds = \int_{S^1} k \left< \frac{\partial N}{\partial s}, T(s) \right> ds \]

\[ = -\int_{S^1} k < N, \frac{\partial T}{\partial s} > ds = -\int_{S^1} k < N, \frac{\partial T}{\partial r} > dr \]

\[ = -\int_{S^1} k^2 dr \]
Curve Shortening Flow
The theorem (Gage-Hamilton, Grayson) states:

The curve shortening flow shrinks any simple closed plane curve $C$ to a point, and $C$ becomes round as it evolves, in the sense that the ratio of the inscribed radius to the circumscribed radius approaches 1.
Curve Shortening Flow
“Proof”

**Step 1.** Any simple closed plane curve evolves in finite time to a convex curve (Grayson)

**Step 2.** Any convex plane curve evolves in finite time to a point, and the curve becomes round (Gage-Hamilton)
Evolution of curvature under the flow

\[ \frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3 \]
Strong Maximum Principle for Heat Equation

\[ \frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3 \]

\[ k(s, 0) \geq 0 \implies k(s, t) > 0, \text{ for all } s \in S^1, t > 0. \]
Strong Maximum Principle for Heat Equation

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Strong Maximum Principle for Heat Equation

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Curve Shortening Flow
Curve Shortening Flow

\[ A_1'(t) = - \int_{C_1} k \, dr = \alpha(t) - 3\pi \]

\[ A_2'(t) = - \int_{C_2} k \, dr = -\alpha(t) - \pi \]

\[ A_1(0) > 3A_2(0) \implies A_2(t) = 0 \text{ before } t = \frac{A_2(0)}{\pi} \]
Curve Shortening Flow

\[ \int_C k \, ds = 2\pi \]
\[ \int_C k \, ds = 2\pi \]

\[ 0 \leq \hat{\tau} < 2\pi \]
Curve Shortening Flow

\[ \int_C k \, ds = 2\pi \]

\[ \hat{r}(s_0) = 0 \]
Curve Shortening Flow

\[ \int_C k \, ds = 2\pi \]
Curve Shortening Flow

\[ \int_C k \, ds = 2\pi \]

\[ \hat{t}(s) \]

\[ \hat{t}(s_0) = 0 \]
\[ \int_C k \, ds = 2\pi \]

As \( s \to s_0^-, \hat{\tau}(s) \to 2\pi \)
Curve Shortening Flow

$$\int_C k \, ds = 2\pi$$

As \( s \to s_0^- \), \( \hat{r}(s) \to 2\pi \)

\( \hat{r} \) is not continuous!
\[ \int_C k \, ds = 2\pi \]

There exists a differentiable function \( \tau \) such that \( \tau(s) \equiv \hat{\tau}(s) \mod 2\pi \)}
There exists a differentiable function $\tau$ such that

$$\tau(s) \equiv \hat{r}(s) \pmod{2\pi}$$

$$T(s) = (\cos \tau(s), \sin \tau(s))$$
\[ \int_C k \, ds = 2\pi \]

\[ T(s) = (\cos \tau(s), \sin \tau(s)) \]

\[ N(s) = (-\sin \tau(s), \cos \tau(s)) \]
\[ \int_C k \, ds = 2\pi \]

Curve Shortening Flow

\[ T(s) = (\cos \tau(s), \sin \tau(s)) \]

\[ N(s) = (-\sin \tau(s), \cos \tau(s)) \]

\[ k(s) = \langle \frac{dT}{ds}, N(s) \rangle \]

\[ = \tau'(s) \]
\[
\int_C k \, ds = 2\pi \\
k(s) = \tau'(s) \\
\int_C k \, ds = \tau(\ell) - \tau(0)
\]
\[ \int_C k \, ds = 2\pi \]

\[ k(s) = \tau'(s) \]

\[ \int_C k \, ds = \tau(\ell) - \tau(0) \]
\[ \int_C k \, ds = 2\pi \]

\[ k(s) = \tau'(s) \]

\[ \int_C k \, ds = \tau(\ell) - \tau(0) = 2\pi \]


**Curve Shortening Flow**

\[ \int_C k \, ds = 2\pi \]
Curve Shortening Flow

\[ \int_C k \, ds = 2\pi \]

where

\[ -\pi < j(s_0) < 0 \text{ or } 0 < j(s_0) < \pi \]

depending on whether \( \{ T(s_0^-), T(s_0^+) \} \) has positive orientation or negative orientation.
Curve Shortening Flow

\[ \int_{C} k \, ds = 2\pi \]

\[ \int_{\Gamma} k \, ds = 2\pi - j(s_0) \]

where

\[ -\pi < j(s_0) < 0 \text{ or } 0 < j(s_0) < \pi \]

depending on whether \( \{T(s_0^-), T(s_0^+)\} \) has positive orientation or negative orientation.
Curve Shortening Flow

\[ A'_1(t) = - \int_{C_1} k \, dr = \alpha(t) - 3\pi \]

\[ A'_2(t) = - \int_{C_2} k \, dr = -\alpha(t) - \pi \]

\[ A_1(0) > 3A_2(0) \implies A_2(t) = 0 \text{ before } t = \frac{A_2(0)}{\pi} \]
Mean Curvature Flow

Let $M^n$ be an embedded (immersed) surface in $\mathbb{R}^{n+1}$.

\[
\frac{\partial C}{\partial t}(x, t) = H(x, t)N(x, t) \quad C(x, 0) = x \in M^n
\]
Any 3-dimensional simply connected, connected and closed manifold is diffeomorphic to the 3-dimensional sphere.
Poincare Conjecture
Any 3-dimensional simply connected, connected and closed manifold is diffeomorphic to the 3-dimensional sphere.
Recognize a sphere

$$( M^n, g )$$

g(x): an inner product structure on $T_x M^n$
Recognize a sphere

\[ \left( M^n, g \right) \text{ Riemannian Manifold} \]

g(x): an inner product structure on \( T_x M^n \)
Recognize a sphere

$S$: 2-dim subspace of $T_x M^n$

Sectional Curvature of $S = \text{the Gaussian Curvature of } \sum^2$ at $x$
Recognize a sphere

$S$: 2-dim subspace of $T_xM^n$

$K(S) = \text{the sectional curvature of } S = K(u,v)$
Recognize a sphere

\[ S^n : \text{the unit sphere in } \mathbb{R}^{n+1} \text{ with the induced Euclidean metric.} \]

The sectional curvature of any 2–dim subspace of the tangent space at any point on the sphere = 1.
Poincare Conjecture

Recognize a sphere

\( S^n \): the unit sphere in \( \mathbb{R}^{n+1} \) with the induced Euclidean metric.

\[ K \equiv 1 \]
Poincare Conjecture

Recognize a sphere

\((M^n, g)\) : simply connected and connected Riemannian manifold

\[ K \equiv 1 \implies M^n \text{ isometric (diffeomorphic) to } S^n \]
\[ M^3 : \text{simply connected, connected and closed manifold} \]

Construct a metric \( g \) on \( M^3 \) such that \( K \equiv 1 \)
$M^3$: simply connected, connected and closed manifold

Put a Riemannian metric, say $g_0$, on $M^3$
$M^3$ : simply connected, connected and closed manifold

Put a Riemannian metric, say $g_0$, on $M^3$
$\mathcal{M}^3$: simply connected, connected and closed manifold

Put a Riemannian metric, say $g_0$, on $\mathcal{M}^3$
\( M^3 \) : simply connected, connected and closed manifold

Put a Riemannian metric, say \( g_0 \), on \( M^3 \)

Evolve \( g_0 \) in a clever way so that it will become round
Let $u \in T_x M^n$ be a unit vector. $\{u, u_2, \ldots, u_n\}$ orthonormal basis for $T_x M^n$. Define the Ricci curvature in the direction of $u$:

$$Ric(u) = K(u, u_2) + \ldots + K(u, u_n)$$
Let $u \in T_x M^n$ be a unit vector. $\{u, u_2, \ldots, u_n\}$ orthonormal basis for $T_x M^n$. Define the Ricci curvature in the direction of $u$:

$$\text{Ric}(u) = K(u, u_2) + \ldots + K(u, u_n)$$

Extend Ric to a linear map on $T_x M^n$. 
Let $u \in T_xM^n$ be a unit vector. \{u, u_2, \ldots, u_n\} orthonormal basis for $T_xM^n$. Define the Ricci curvature in the direction of $u$:

$$\text{Ric}(u) = K(u, u_2) + \ldots + K(u, u_n)$$

Extend $\text{Ric}$ to a linear map on $T_xM^n$.

Extend $\text{Ric}$ to a bi-linear map on $T_xM^n$:

$$\text{Ric}(u, v) = \frac{1}{2}[\text{Ric}(u + v) - \text{Ric}(u) - \text{Ric}(v)]$$
Let \((M^n, g_0)\) be a closed Riemannian manifold.

Ricci flow on \((M^n, g_0)\) is a one-parameter family of metrics \(g(t)\) such that

\[
\frac{\partial g}{\partial t} = -2\operatorname{Ric}(g(t))
\]

\(g(0) = g_0\)
Let $(S^n, g_0)$ be the standard unit sphere.
Define
\[ g(t) = (1 - 2(n - 1)t)g_0 \]
Then
\[ \frac{\partial g}{\partial t} = -2\text{Ric}(g(t)) \]
\[ g(0) = g_0 \]
Theorem (Short Time Existence and Uniqueness)

Let \((M^n, g_0)\) be a closed Riemannian manifold. There exists a \(T > 0\) and a unique family of Riemannian metric \(g(t)\) for \(t \in [0, T)\) such that

\[
\frac{\partial g}{\partial t} = -2\text{Ric}(g(t))
\]

\(g(0) = g_0\)
Ricci Flow

\[ \frac{\partial g}{\partial t} = -2\text{Ric}(g(t)) \]
Ricci Flow
Theorem (Hamilton)

If a simply connected, connected and closed \((M^3, g_0)\) has positive Ricci curvature, then the Ricci flow evolves \(g_0\) to a round metric.
Ricci Flow

\[
\frac{\partial g}{\partial t} = -2\text{Ric}(g(t))
\]
Ricci Flow
Ricci Flow
Ricci Flow
Ricci Flow
Ricci Flow
Ricci Flow

degenerate neck pinch
A gradient Ricci soliton is a Riemannian manifold \((M, g)\) together with a smooth function \(f\) such that

\[ \text{Ric} + \text{Hess} f = \lambda g, \]

where \(\lambda\) is a constant. It is called shrinking, steady and expanding when \(\lambda > 0\), \(\lambda = 0\) and \(\lambda < 0\) respectively.
Shrinking Gradient Ricci Soliton

\[ \text{Ric} + \text{Hess} f = \frac{1}{2} g \]
Shrinking Gradient Ricci Soliton

\[ \text{Ric} + \text{Hess} f = \frac{1}{2} g \]

Define \( \Phi_t : M \to M \) with \( \Phi_0 = \text{Id} \) and \( \frac{\partial \Phi}{\partial t} = \nabla f \)

\[ G(t) = (1 - t) \Phi_{-\ln(1-t)}^* g \]

\[ \frac{\partial G}{\partial t} = -2\text{Ric}(G(t)) \]

\[ G(0) = g \]
Theorem (Perelman)

An open 3-dimensional shrinking gradient Ricci soliton with bounded nonnegative sectional curvature is a quotient of $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$. 
Theorem (Perelman)

Any 3-dimensional simply connected, connected and closed manifold is diffeomorphic to the 3-dimensional sphere
Classifications of Shrinking Gradient Ricci Soliton

- Zero Weyl tensor
  
  Petersen and Wylie
  
  Cao, Wang and Zhu

- Harmonic Weyl tensor
  
  Fernández-López and Garcia-Rio

- Nonnegative sectional curvature and constant scalar curvature
  
  Petersen and Wylie
Theorem (C.)

Let \((M, g, f)\) be a complete non-compact shrinking gradient Ricci soliton with bounded nonnegative sectional curvature. Assume that there exists \(\delta > 0\) such that

\[
\int_M e^{\delta f} |\nabla R| \, dvol_g < \infty.
\]

Then \((M^n, g)\) is isometric to \(N \times_{\Gamma} \mathbb{R}^m\), where \(N\) is a compact Einstein manifold.
“Proof”

Step 1. Show the scalar curvature is constant

Step 2. Petersen and Wylie
Acknowledgement

• https://en.wikipedia.org

• http://www.numberphile.com

• http://www.ams.org

• http://homepages.warwick.ac.uk/~maseq/topping_RF_mar06.pdf
Thank You
\[ W(g, f, \tau) = \int_M \left[ \tau \left( |\nabla f|^2 + R \right) + f - n \right] (4\pi\tau)^{-n/2} e^{-f} dV \]
\[ = (4\pi\tau)^{-n/2} \tau \mathcal{F}(g, f) + (4\pi\tau)^{-n/2} \int_M (f - n) e^{-f} dV, \]

\[ \tilde{V}_{(p,0)}(\tau) := \int_M (4\pi\tau)^{-n/2} e^{-\xi(p,0)(q,\tau)} d\mu_{g(\tau)}(q) \]