Fractional Calculus and Smallest Eigenvalues

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In order to talk about fractional calculus, we need to first mention the Gamma function.
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\begin{align*}
\Gamma(x) &= \int_0^\infty t^{x-1} e^{-t} \, dt, \\
\Gamma(x+1) &= x\Gamma(x), \\
\Gamma(n+1) &= n!.
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**Definition**

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$$\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} \, dt.$$ 

Here are the important properties of the Gamma function.

1. For each $x \in (0, \infty)$, $\Gamma(x + 1) = x\Gamma(x)$.
2. For $n \in \mathbb{N}$, $\Gamma(n + 1) = n!$. 
The Gamma Function

Historical Background

The Fractional Integral and Derivative

Examples

Smallest Eigenvalues

Fractional Calculus and Smallest Eigenvalues
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In 1695, after Leibniz first invented the notation \( \frac{d^n y}{dx^n} \), L’Hôpital wrote to Leibniz and asked him, ”What if \( n = 1/2?\)” Leibniz responded, ”It leads to a paradox, from which one day useful consequences will be drawn.”
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\frac{d^n}{dx^n}(x^m) = m \cdot (m - 1) \cdot (m - 2) \cdots (m - (n + 1))x^{m-n}
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He then generalized the formula for all $n$, obtaining

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He then generalized the formula for all $n$, obtaining

$$\frac{d^n}{dx^n}(x^m) = \frac{\Gamma(m + 1)}{\Gamma(m - n + 1)}x^{m-n}.$$ 

Using this definition, he was able to show $\frac{d^{1/2}}{dx^{1/2}}(x) = \frac{2\sqrt{x}}{\sqrt{\pi}}$. 

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Fractional calculus finds use in many fields of science and engineering, including fluid flow, electrical networks, and probability.
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\[
I_n^c f(x) = \int_c^x \int_c^{x_1} \cdots \int_c^{x_{n-1}} f(t) dt \, dx_{n-1} \cdots dx_2 \, dx_1
\]

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= \frac{1}{(n-1)!} \int_c^x (x - t)^{n-1} f(t) dt.
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$$= \frac{1}{(n-1)!} \int_c^x (x - t)^{n-1} f(t) dt.$$

For $n = 2$,

$$I_c^2 f(x) = \int_c^x \int_c^{x_1} f(t) dt \, dx_1$$

$$= \int_c^x \int_t^x f(t) dx_1 \, dt$$

$$= \int_c^x (x - t) f(t) dt.$$
For $n = 3$, if we use the previous result,

$$I_3^c f(x) = \int_c^x \int_c^{x_1} \int_c^{x_2} f(t) dt \, dx_2 \, dx_1$$

$$= \int_c^x \left[ \int_c^{x_1} \int_c^{x_2} f(t) dt \, dx_2 \right] \, dx_1$$

$$= \int_c^x \left[ \int_c^{x_1} (x_1 - t)f(t) dt \right] \, dx_1$$

$$= \int_c^x \int_t^x (x_1 - t)f(t) \, dx_1 \, dt$$

$$= \int_c^x f(t) \frac{(x - t)^2}{2} dt.$$
If we continue in this fashion, we obtain

\[ I^n_{c+} f(x) = \frac{1}{(n-1)!} \int_c^x (x - t)^{n-1} f(t) dt \]

\[ = \frac{1}{\Gamma(n)} \int_c^x (x - t)^{n-1} f(t) dt. \]
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**Definition**

For a function $f(x)$ defined on $(c, \infty)$, define the Riemann-Liouville fractional integral of order $\alpha > 0$ of $f(x)$ by

$$I_{c+}^\alpha f(x) \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t) dt,$$

provided the integral exists.
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**Definition**

Let \( \alpha > 0 \). Let \( n = [\alpha] + 1 \). For a function \( f(x) \) defined on \((c, \infty)\), define the Riemann-Liouville fractional derivative of order \( \alpha \) of \( f(x) \) by

\[
D^\alpha_{c+} f(x) = \frac{d^n}{dx^n} I^{n-\alpha}_{c+} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{c}^{x} (x-t)^{n-\alpha-1} f(t) dt,
\]

provided the integral exists.
Let $k > -1$ and $n = \lfloor \alpha \rfloor + 1$. Then

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For example, if $\alpha = 1/2$, using the fact that $\Gamma(1/2) = \sqrt{\pi}$,

$$D_0^{1/2} 1 = \frac{\Gamma(1)}{\Gamma(1/2)} x^{-1/2} = \frac{1}{\sqrt{\pi}x}.$$
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Note that if we try to take the $1/2$ derivative of $\frac{1}{\sqrt{\pi}x}$, we obtain

$$D_{0+}^{1/2} \frac{1}{\sqrt{\pi}x} = \frac{\Gamma(1/2)}{\Gamma(0)} \frac{1}{x\sqrt{\pi}}.$$
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But $\Gamma(0)$ is not defined. However, since $\lim_{x \to 0} |\Gamma(x)| = \infty$, we can define $\frac{1}{\Gamma(0)} = 0$. 
Let $k > -1$ and $n = \lfloor \alpha \rfloor + 1$. Then

$$D^\alpha_0 x^k = \frac{\Gamma(k + 1)}{\Gamma(k - \alpha + 1)} x^{k-\alpha}$$

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For $n \in \mathbb{N}$,

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$$D^{1/2}_0 e^x = D^{1/2}_0 (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)$$

$$= \frac{\Gamma(1)}{\Gamma(1/2)} x^{-1/2} + \frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2} + \frac{\Gamma(3)}{2!\Gamma(5/2)} x^{3/2} + \frac{\Gamma(4)}{3!\Gamma(7/2)} x^{5/2} + \cdots$$

$$= \frac{1}{\sqrt{\pi}x} (1 + 2x + \frac{4}{3}x^2 + \frac{8}{15}x^3 + \cdots)$$

$$\neq e^x.$$
But for \( c = -\infty \), we have

\[
I^{1/2}_{-\infty} e^x = \frac{1}{\Gamma(1/2)} \int_{-\infty}^{\infty} (x - t)^{-1/2} e^t dt
\]

\[
= \frac{1}{\sqrt{\pi}} \sqrt{\pi} e^x
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\[
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Thus \( D_{-\infty}^{1/2} e^x = \frac{d}{dx} I_{-\infty}^{1/2} e^x = \frac{d}{dx} e^x = e^x \).
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Thus $D_{-\infty}^{1/2} e^x = \frac{d}{dx} I_{-\infty}^{1/2} e^x = \frac{d}{dx} e^x = e^x$.

In general, for $c = -\infty$, $D_{-\infty}^{\alpha} e^{ax} = a^{\alpha} e^{ax}$. 
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\[ f'(x) = \cos x = \sin(x + \pi/2), \]
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and in general, \( f^{(n)}(x) = \sin(x + (n\pi)/2) \).
Let’s consider one last example. For \( f(x) = \sin x \),
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  f^{(n)}(x) &= \sin(x + (n\pi)/2).
\end{align*}
\]
Similarly, for \( g(x) = \cos x \), \( g^{(n)}(x) = \cos(x + (n\pi)/2) \). In general,
\[
\begin{align*}
  D^\alpha_{-\infty} \sin x &= \sin(x + (\alpha\pi)/2) \quad \text{and} \\
  D^\alpha_{-\infty} \cos x &= \cos(x + (\alpha\pi)/2).
\end{align*}
\]
We consider the comparison of smallest eigenvalues for the eigenvalue problems

\[ D_0^\alpha u + \lambda_1 p(t)u = 0, \quad 0 < t < 1, \quad (1) \]
\[ D_0^\alpha u + \lambda_2 q(t)u = 0, \quad 0 < t < 1, \quad (2) \]
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\[ D_0^\alpha u + \lambda_2 q(t) u = 0, \quad 0 < t < 1, \] \tag{2}

satisfying the boundary conditions

\[ u(0) = u(1) = 0, \] \tag{3}
We consider the comparison of smallest eigenvalues for the eigenvalue problems

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\end{align*}

satisfying the boundary conditions

\begin{equation}
\begin{split}
u(0) &= u(1) = 0,
\end{split}
\end{equation}

where \(1 < \alpha \leq 2\) is a real number, \(D_0^\alpha\) is the standard Riemann-Liouville derivative, and \(p(t)\) and \(q(t)\) are continuous nonnegative functions on \([0, 1]\), where neither \(p(t)\) nor \(q(t)\) vanishes identically on any compact subinterval of \([0, 1]\).
Definition

Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided

(i) $\alpha u + \beta v \in \mathcal{P}$, for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and

(ii) $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ implies $u = 0$. 
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Definition

A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^\circ$, of $\mathcal{P}$, is nonempty. A cone $\mathcal{P}$ is reproducing if $\mathcal{B} = \mathcal{P} - \mathcal{P}$; i.e., given $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$. 
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Krasnosel’skii showed that every solid cone is reproducing.
Definition

Let \( \mathcal{P} \) be a cone in a real Banach space \( \mathcal{B} \). If \( u, v \in \mathcal{B} \), \( u \leq v \) with respect to \( \mathcal{P} \) if \( v - u \in \mathcal{P} \). If both \( M, N : \mathcal{B} \to \mathcal{B} \) are bounded linear operators, \( M \leq N \) with respect to \( \mathcal{P} \) if \( Mu \leq Nu \) for all \( u \in \mathcal{P} \).
Definition

Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u, v \in \mathcal{B}, u \leq v$ with respect to $\mathcal{P}$ if $v - u \in \mathcal{P}$. If both $M, N : \mathcal{B} \to \mathcal{B}$ are bounded linear operators, $M \leq N$ with respect to $\mathcal{P}$ if $Mu \leq Nu$ for all $u \in \mathcal{P}$.

Definition

A bounded linear operator $M : \mathcal{B} \to \mathcal{B}$ is $u_0$-positive with respect to $\mathcal{P}$ if there exists $0 \neq u_0 \in \mathcal{P}$ such that for each $0 \neq u \in \mathcal{P}$, there exist $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \leq Mu \leq k_2 u_0$ with respect to $\mathcal{P}$. 
Lemma

Let $\mathcal{B}$ be a Banach space over the reals, and let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $M : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $M : \mathcal{P}\{0\} \to \mathcal{P}^\circ$, then $M$ is $u_0$-positive with respect to $\mathcal{P}$.
Theorem (Krasnosel’skii)

Let $\mathcal{B}$ be a real Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $L : \mathcal{B} \to \mathcal{B}$ be a compact, $u_0$-positive, linear operator. Then $L$ has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.
Theorem (Krasnosel’skii)

Let \( \mathcal{B} \) be a real Banach space and let \( \mathcal{P} \subset \mathcal{B} \) be a reproducing cone. Let \( L : \mathcal{B} \rightarrow \mathcal{B} \) be a compact, \( u_0 \)-positive, linear operator. Then \( L \) has an essentially unique eigenvector in \( \mathcal{P} \), and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.

Theorem (Krasnosel’skii)

Let \( \mathcal{B} \) be a real Banach space and \( \mathcal{P} \subset \mathcal{B} \) be a cone. Let both \( M, N : \mathcal{B} \rightarrow \mathcal{B} \) be bounded, linear operators and assume that at least one of the operators is \( u_0 \)-positive. If \( M \leq N \), \( My_1 \geq \lambda_1 y_1 \) for some \( y_1 \in \mathcal{P} \) and some \( \lambda_1 > 0 \), and \( Ny_2 \leq \lambda_2 y_2 \) for some \( y_2 \in \mathcal{P} \) and some \( \lambda_2 > 0 \), then \( \lambda_1 \leq \lambda_2 \). Furthermore, \( \lambda_1 = \lambda_2 \) implies \( y_1 \) is a scalar multiple of \( y_2 \).
We derive comparison results for these eigenvalue problems by applying the previous theorems mentioned. To do this, we will define integral operators whose kernel is the Green’s function of $-D_{0+}^{\alpha} u(t) = 0$, (3), which is given by

$$G(t, s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

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So $u(t)$ solves (1),(3) if and only if

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So \( u(t) \) solves (1),(3) if and only if

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u(t) = \lambda_1 \int_0^1 G(t, s)p(s)u(s)ds,
\]

and \( u(t) \) solves (2),(3) if and only if

\[
u(t) = \lambda_2 \int_0^1 G(t, s)q(s)u(s)ds.
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$$u(t) = \lambda_2 \int_0^1 G(t, s)q(s)u(s)ds.$$
To apply Krasnosel’skii’s theorems, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. 
To apply Krasnosel’skii’s theorems, we need to define a Banach space $B$ and a cone $P \subset B$. Define the Banach space $B$ by

$$B = \{ u : u = t^{\alpha-1}v, \, v \in C^1[0,1], \, v(1) = 0 \},$$

with the norm

$$||u|| = \sup_{t \in [0,1]} |v'(t)|.$$
To apply Krasnosel’skii’s theorems, we need to define a Banach space $B$ and a cone $\mathcal{P} \subset B$. Define the Banach space $B$ by

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Define the cone

$$\mathcal{P} = \{ u \in B | u(t) \geq 0 \text{ for } t \in [0, 1] \}.$$
Lemma

*The cone $\mathcal{P}$ is solid in $B$ and hence reproducing.*
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Proof: Define

$$\Omega := \{u \in B \mid u(t) > 0 \text{ for } t \in (0, 1), \quad v(0) > 0, \quad v'(1) < 0\}.$$ (5)
Lemma

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$\Omega \subset \mathcal{P}^\circ$. So $\mathcal{P}$ is solid and hence reproducing.
Define the compact linear operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mu(t) = \int_{0}^{1} G(t, s)p(s)u(s)ds$$  \hspace{1cm} (6)$$

and

$$Nu(t) = \int_{0}^{1} G(t, s)q(s)u(s) \, ds.$$  \hspace{1cm} (7)$$
Lemma

The bounded linear operators $M$ and $N$ are $u_0$-positive with respect to $\mathcal{P}$.
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Proof: We show $M : \mathcal{P} \setminus \{0\} \rightarrow \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(t) \geq 0$. 

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Lemma

The bounded linear operators $M$ and $N$ are $u_0$-positive with respect to $\mathcal{P}$.

Proof: We show $M : \mathcal{P} \setminus \{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(t) \geq 0$. Then since $G(t, s) \geq 0$ on $[0, 1] \times [0, 1]$ and $p(t) \geq 0$ on $[0, 1]$,

$$Mu(t) = \int_0^1 G(t, s)p(s)u(s)ds \geq 0,$$

for $0 \leq t \leq 1$. So $M : \mathcal{P} \to \mathcal{P}$. 
Now let $u \in \mathcal{P}\{0\}$. 
Now let $u \in \mathcal{P}\{0\}$. So there exists a compact interval $[\alpha, \beta] \subset [0, 1]$ such that $u(t) > 0$ and $p(t) > 0$ for all $t \in [\alpha, \beta]$. Then, since $G(t, s) > 0$ on $(0, 1) \times (0, 1)$,

$$Mu(t) = \int_{0}^{1} G(t, s)p(s)u(s)ds$$

$$\geq \int_{\alpha}^{\beta} G(t, s)p(s)u(s)ds$$

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for $0 < t < 1$. 

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Now

\[ M_u(t) = t^{\alpha-1} \left( \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)\,ds \right) \]

\[ - t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)\,ds \].
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\[ Mu(t) = t^{\alpha-1} \left( \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} p(s) u(s) ds \right) \]

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Let

\[ v(t) = \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} p(s) u(s) ds - t^{1-\alpha} \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} p(s) u(s) ds . \]
So \( v(0) = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) u(s) ds > 0 \)
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\[
v'(1) = (\alpha - 1) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds
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\[
- (\alpha - 1) \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} p(s) u(s) ds < 0.
\]

So \( M : \mathcal{P}\setminus\{0\} \to \Omega \subset \mathcal{P}^\circ. \)
Notice that

\[ \Lambda u = Mu = \int_0^1 G(t, s)p(s)u(s)ds, \]

if and only if

\[ D_{\alpha}^0 u(t) + \Lambda p(t)u(t) = 0, \]

with

\[ u(0) = u(1) = 0. \]
Notice that

$$\Lambda u = Mu = \int_{0}^{1} G(t, s)p(s)u(s)ds,$$

if and only if

$$u(t) = \frac{1}{\Lambda} \int_{0}^{1} G(t, s)p(s)u(s)ds,$$

So the eigenvalues of (1),(3) are reciprocals of eigenvalues of $M$, and conversely.

Similarly, eigenvalues of (2),(3) are reciprocals of eigenvalues of $N$, and conversely.
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$$D_{0+}^\alpha u(t) + \frac{1}{\Lambda} p(t)u(t) = 0, \quad 0 < t < 1,$$

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So the eigenvalues of (1),(3) are reciprocals of eigenvalues of \( M \), and conversely. Similarly, eigenvalues of (2),(3) are reciprocals of eigenvalues of \( N \), and conversely.
Theorem

Let $B$, $P$, $M$, and $N$ be defined as earlier. Then $M$ (and $N$) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $P^\circ$. 
Theorem

Let $B$, $P$, $M$, and $N$ be defined as earlier. Then $M$ (and $N$) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $P^\circ$.

Theorem

Let $B$, $P$, $M$, and $N$ be defined as earlier. Let $p(t) \leq q(t)$ on $[0,1]$. Let $\Lambda_1$ and $\Lambda_2$ be the eigenvalues defined in the previous theorem associated with $M$ and $N$, respectively, with the essentially unique eigenvectors $u_1$ and $u_2 \in P^\circ$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(t) = q(t)$ on $[0,1]$. 
The following theorem is an immediate consequence of the relationship between the eigenvalues of $M$ and (1),(3), and the eigenvalues of $N$ and (2),(3), and the previous two theorems.
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**Theorem**

Assume the hypotheses of the previous theorem. Then there exists smallest positive eigenvalues $\lambda_1$ and $\lambda_2$ of (1),(3) and (2),(3), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to $\lambda_1$ and $\lambda_2$ may be chosen to belong to $P^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(t) = q(t)$ for all $t \in [0, 1]$. 