Fractional Calculus and Smallest Eigenvalues

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Nova Southeastern Mathematics Colloquium Series
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$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$ 

Here are the important properties of the Gamma function.

1. For each $x \in (0, \infty)$, $\Gamma(x + 1) = x\Gamma(x)$.
2. For $n \in \mathbb{N}$, $\Gamma(n + 1) = n!$. 
The Gamma Function

Historical Background

The Fractional Integral and Derivative

Examples

Smallest Eigenvalues

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The original question that led to the name fractional calculus was: Can the meaning of a derivative of integer order \( \frac{d^n y}{dx^n} \) be extended to have meaning when \( n \) is a fraction?
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In 1695, after Leibniz first invented the notation \( \frac{d^n y}{dx^n} \), L’Hôpital wrote to Leibniz and asked him, ”What if \( n = 1/2?\)” Leibniz responded, ”It leads to a paradox, from which one day useful consequences will be drawn.”
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$$\frac{d^n}{dx^n}(x^m) = m \cdot (m - 1) \cdot (m - 2) \cdots (m - (n + 1))x^{m-n}$$

$$= \frac{m!}{(m-n)!}x^{m-n}, \quad n \in \mathbb{N}, \quad m \geq n.$$
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He then generalized the formula for all $n$, obtaining

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Using this definition, he was able to show $\frac{d^{1/2}}{dx^{1/2}}(x) = \frac{2\sqrt{x}}{\sqrt{\pi}}$. 

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Fractional calculus finds use in many fields of science and engineering, including fluid flow, electrical networks, and probability.
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$$I^n_{c+} f(x) = \int_c^x \int_c^{x_1} \cdots \int_c^{x_{n-1}} f(t) \, dt \, dx_{n-1} \cdots dx_2 \, dx_1$$

$$= \frac{1}{(n-1)!} \int_c^x (x - t)^{n-1} f(t) \, dt.$$
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$$= \frac{1}{(n-1)!} \int_c^x (x-t)^{n-1} f(t) \, dt.$$ 

For $n = 2$,

$$I^2_{c^+} f(x) = \int_c^x \int_c^{x_1} f(t) \, dt \, dx_1$$

$$= \int_c^x \int_t^x f(t) \, dx_1 \, dt$$

$$= \int_c^x (x-t) f(t) \, dt$$
For $n = 3$, if we use the previous result, 

$$I_{c^+}^3 f(x) = \int_c^x \int_c^{x_1} \int_c^{x_2} f(t) dt \, dx_2 \, dx_1$$

$$= \int_c^x \left[ \int_c^{x_1} \int_c^{x_2} f(t) dt \, dx_2 \right] \, dx_1$$

$$= \int_c^x \left[ \int_c^{x_1} (x_1 - t)f(t) dt \right] \, dx_1$$

$$= \int_c^x \int_t^x (x_1 - t)f(t) dx_1 \, dt$$

$$= \int_c^x f(t) \frac{(x - t)^2}{2} dt.$$
If we continue in this fashion, we obtain

\[ I_c^n f(x) = \frac{1}{(n-1)!} \int_c^x (x-t)^{n-1} f(t) dt \]

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**Definition**

For a function \( f(x) \) defined on \((c, \infty)\), define the Riemann-Liouville fractional integral of order \( \alpha > 0 \) of \( f(x) \) by

\[ I_{c+}^\alpha f(x) \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t) \, dt, \]

provided the integral exists.
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**Definition**

Let $\alpha > 0$. Let $n = [\alpha] + 1$. For a function $f(x)$ defined on $(c, \infty)$, define the Riemann-Liouville fractional derivative of order $\alpha$ of $f(x)$ by

$$D^\alpha_{c^+} f(x) = \frac{d^n}{dx^n} I^{n-\alpha}_{c^+} f(x)$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_c^x (x-t)^{n-\alpha-1} f(t) dt,$$

provided the integral exists.
Let $k > -1$ and $n = \lfloor \alpha \rfloor + 1$. Then

$$D^\alpha_0 x^k = \frac{\Gamma(k + 1)}{\Gamma(k - \alpha + 1)} x^{k-\alpha}$$
Let $k > -1$ and $n = [\alpha] + 1$. Then

$$D_{0+}^{\alpha} x^k = \frac{\Gamma(k + 1)}{\Gamma(k - \alpha + 1)} x^{k-\alpha}$$

For example, if $\alpha = 1/2$, using the fact that $\Gamma(1/2) = \sqrt{\pi}$,

$$D_{0+}^{1/2} 1 = \frac{\Gamma(1)}{\Gamma(1/2)} x^{-1/2} = \frac{1}{\sqrt{\pi x}}.$$
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Note that if we try to take the 1/2 derivative of \( \frac{1}{\sqrt{\pi x}} \), we obtain

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D_0^{1/2} \frac{1}{\sqrt{\pi x}} = \frac{\Gamma(1/2)}{\Gamma(0)} \frac{1}{x\sqrt{\pi}}
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Let $k > -1$ and $n = [\alpha] + 1$. Then

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Note that if we try to take the $1/2$ derivative of $\frac{1}{\sqrt{\pi x}}$, we obtain

$$D_{0+}^{1/2} \frac{1}{\sqrt{\pi x}} = \frac{\Gamma(1/2)}{\Gamma(0)} \frac{1}{x\sqrt{\pi}}$$

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Let $k > -1$ and $n = [\alpha] + 1$. Then

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For example, for \( \alpha = 1/2 \), let’s look at the standard
Riemann-Liouville fractional derivative of \( e^x \) (\( c = 0 \)). Using the
Taylor Series of \( e^x \), we obtain
\[
D_{0^+}^{1/2} e^x = D_{0^+}^{1/2} \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)
= \frac{\Gamma(1)}{\Gamma(1/2)} x^{-1/2} + \frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2} + \frac{\Gamma(3)}{2!\Gamma(5/2)} x^{3/2} + \frac{\Gamma(4)}{3!\Gamma(7/2)} x^{5/2} + \cdots
= \frac{1}{\sqrt{\pi} x} \left(1 + 2x + \frac{4}{3} x^2 + \frac{8}{15} x^3 + \cdots \right)
\neq e^x.
\]
But for \( c = -\infty \), we have

\[
I_{-\infty}^{1/2} e^x = \frac{1}{\Gamma(1/2)} \int_{-\infty}^{x} (x - t)^{-1/2} e^t \, dt
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Thus $D_{-\infty}^{1/2} e^x = \frac{d}{dx} I_{-\infty}^{1/2} e^x = \frac{d}{dx} e^x = e^x$.

In general, for $c = -\infty$, $D_{-\infty}^{\alpha} e^{ax} = a^\alpha e^{ax}$. 
Let’s consider one last example.

For \( f(x) = \sin x \),
\[ f'(x) = \cos x = \sin(x + \pi/2), \]
\[ f''(x) = -\sin x = \sin(x + (2\pi)/2), \]
\[ f'''(x) = -\cos x = \sin(x + (3\pi)/2), \]
and in general,
\[ f^{(n)}(x) = \sin(x + (n\pi)/2). \]

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Similarly, for \( g(x) = \cos x \), \( g^{(n)}(x) = \cos(x + (n\pi)/2). \) In general,
\[ D_{-\infty}^{\alpha} \sin x = \sin(x + (\alpha\pi)/2) \] and \( D_{-\infty}^{\alpha} \cos x = \cos(x + (\alpha\pi)/2). \)
We consider the comparison of smallest eigenvalues for the eigenvalue problems

\begin{align*}
D_0^\alpha u + \lambda_1 p(t)u &= 0, \quad 0 < t < 1, \quad (1) \\
D_0^\alpha u + \lambda_2 q(t)u &= 0, \quad 0 < t < 1, \quad (2)
\end{align*}
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\end{align*}

(1) (2)

satisfying the boundary conditions

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 u(0) = u(1) = 0,
\end{equation}

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satisfying the boundary conditions

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where \( 1 < \alpha \leq 2 \) is a real number, \( D_0^\alpha \) is the standard Riemann-Liouville derivative, and \( p(t) \) and \( q(t) \) are continuous nonnegative functions on \([0, 1]\), where neither \( p(t) \) nor \( q(t) \) vanishes identically on any compact subinterval of \([0, 1]\).
Definition

Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided

(i) $\alpha u + \beta v \in \mathcal{P}$, for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and

(ii) $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ implies $u = 0$. 

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Definition

A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^\circ$, of $\mathcal{P}$, is nonempty. A cone $\mathcal{P}$ is reproducing if $\mathcal{B} = \mathcal{P} - \mathcal{P}$; i.e., given $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$. 
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Krasnosel’skii showed that every solid cone is reproducing.
**Definition**

Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u, v \in \mathcal{B}$, $u \leq v$ with respect to $\mathcal{P}$ if $v - u \in \mathcal{P}$. If both $M, N : \mathcal{B} \rightarrow \mathcal{B}$ are bounded linear operators, $M \leq N$ with respect to $\mathcal{P}$ if $Mu \leq Nu$ for all $u \in \mathcal{P}$. 

Jeffrey T. Neugebauer
Nova Southeastern Mathematics Colloquium Series
Fractional Calculus and Smallest Eigenvalues
Definition

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Definition

A bounded linear operator \( M : \mathcal{B} \rightarrow \mathcal{B} \) is \( u_0 \)-positive with respect to \( \mathcal{P} \) if there exists \( 0 \neq u_0 \in \mathcal{P} \) such that for each \( 0 \neq u \in \mathcal{P} \), there exist \( k_1(u) > 0 \) and \( k_2(u) > 0 \) such that \( k_1 u_0 \leq Mu \leq k_2 u_0 \) with respect to \( \mathcal{P} \).
Lemma

Let $\mathcal{B}$ be a Banach space over the reals, and let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $M : \mathcal{B} \rightarrow \mathcal{B}$ is a linear operator such that $M : \mathcal{P}\backslash\{0\} \rightarrow \mathcal{P}^\circ$, then $M$ is $u_0$-positive with respect to $\mathcal{P}$. 
Theorem (Krasnosel’skii)

Let $B$ be a real Banach space and let $\mathcal{P} \subset B$ be a reproducing cone. Let $L : B \to B$ be a compact, $u_0$-positive, linear operator. Then $L$ has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.
Theorem (Krasnosel’skii)

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Theorem (Krasnosel’skii)

Let $\mathcal{B}$ be a real Banach space and $\mathcal{P} \subset \mathcal{B}$ be a cone. Let both $M, N : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators and assume that at least one of the operators is $u_0$-positive. If $M \leq N$, $My_1 \geq \lambda_1 y_1$ for some $y_1 \in \mathcal{P}$ and some $\lambda_1 > 0$, and $Ny_2 \leq \lambda_2 y_2$ for some $y_2 \in \mathcal{P}$ and some $\lambda_2 > 0$, then $\lambda_1 \leq \lambda_2$. Furthermore, $\lambda_1 = \lambda_2$ implies $y_1$ is a scalar multiple of $y_2$. 
We derive comparison results for these eigenvalue problems by applying the previous theorems mentioned. To do this, we will define integral operators whose kernel is the Green’s function of $-D_0^\alpha u(t) = 0$, (3), which is given by

$$G(t, s) = \begin{cases} 
\frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\
\frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1.
\end{cases}$$

So $u(t)$ solves (1),(3) if and only if $u(t) = \lambda_1 \int_0^1 G(t, s) p(s) u(s) ds$, and $u(t)$ solves (2),(3) if and only if $u(t) = \lambda_2 \int_0^1 G(t, s) q(s) u(s) ds$. 

Jeffrey T. Neugebauer
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Fractional Calculus and Smallest Eigenvalues
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$$u(t) = \lambda_2 \int_0^1 G(t, s)q(s)u(s)ds.$$
We derive comparison results for these eigenvalue problems by applying the previous theorems mentioned. To do this, we will define integral operators whose kernel is the Green’s function of $-D_{0+}^\alpha u(t) = 0$, (3), which is given by

$$G(t, s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

(4)

So $u(t)$ solves (1),(3) if and only if

$$u(t) = \lambda_1 \int_0^1 G(t, s)p(s)u(s)ds,$$

and $u(t)$ solves (2),(3) if and only if

$$u(t) = \lambda_2 \int_0^1 G(t, s)q(s)u(s)ds.$$
To apply Krasnosel’skii’s theorems, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. 
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$$\mathcal{B} = \{ u : u = t^{\alpha - 1} v, \ v \in C^1[0, 1], \ v(1) = 0 \},$$

with the norm

$$\| u \| = \sup_{t \in [0,1]} |v'(t)|.$$
To apply Krasnosel’skii’s theorems, we need to define a Banach space $B$ and a cone $P \subseteq B$. Define the Banach space $B$ by

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B = \{ u : u = t^{\alpha-1}v, \ v \in C^1[0,1], \ v(1) = 0 \},
$$

with the norm

$$
\| u \| = \sup_{t \in [0,1]} |v'(t)|.
$$

Define the cone

$$
P = \{ u \in B \mid u(t) \geq 0 \text{ for } t \in [0,1] \}.
$$
Lemma

The cone $\mathcal{P}$ is solid in $B$ and hence reproducing.
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Proof: Define

$$\Omega := \{ u \in B \mid u(t) > 0 \text{ for } t \in (0, 1), \, v(0) > 0, \, v'(1) < 0 \}. \quad (5)$$
Lemma

The cone $\mathcal{P}$ is solid in $B$ and hence reproducing.

Proof: Define

$$\Omega := \{u \in B \mid u(t) > 0 \text{ for } t \in (0, 1), \nu(0) > 0, \nu'(1) < 0\}. \quad (5)$$

$\Omega \subset \mathcal{P}^\circ$. So $\mathcal{P}$ is solid and hence reproducing.
Define the compact linear operators $M, N : B \to B$ by

$$Mu(t) = \int_0^1 G(t, s)p(s)u(s)ds$$  \hspace{1cm} (6)$$

and

$$Nu(t) = \int_0^1 G(t, s)q(s)u(s) \, ds.$$  \hspace{1cm} (7)$$
Lemma

The bounded linear operators $M$ and $N$ are $u_0$-positive with respect to $P$. 
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**Proof:** We show $M : \mathcal{P}\setminus\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(t) \geq 0$. 

Lemma

The bounded linear operators $M$ and $N$ are $u_0$-positive with respect to $\mathcal{P}$.

Proof: We show $M : \mathcal{P} \setminus \{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(t) \geq 0$. Then since $G(t,s) \geq 0$ on $[0,1] \times [0,1]$ and $p(t) \geq 0$ on $[0,1]$,

$$Mu(t) = \int_0^1 G(t,s)p(s)u(s) ds \geq 0,$$

for $0 \leq t \leq 1$. So $M : \mathcal{P} \to \mathcal{P}$. 
Now let $u \in \mathcal{P}\backslash\{0\}$. 
Now let $u \in \mathcal{P}\{0\}$. So there exists a compact interval $[\alpha, \beta] \subset [0, 1]$ such that $u(t) > 0$ and $p(t) > 0$ for all $t \in [\alpha, \beta]$.

Then, since $G(t, s) > 0$ on $(0, 1) \times (0, 1)$,

$$Mu(t) = \int_0^1 G(t, s)p(s)u(s)ds$$

$$\geq \int_\alpha^\beta G(t, s)p(s)u(s)ds$$

$$> 0,$$

for $0 < t < 1$. 
Now let $u \in \mathcal{P}\{0\}$. So there exists a compact interval $[\alpha, \beta] \subset [0, 1]$ such that $u(t) > 0$ and $p(t) > 0$ for all $t \in [\alpha, \beta]$. Then, since $G(t, s) > 0$ on $(0, 1) \times (0, 1)$,

$$M u(t) = \int_0^1 G(t, s)p(s)u(s)ds$$

$$\geq \int_{\alpha}^{\beta} G(t, s)p(s)u(s)ds$$

$$> 0,$$

for $0 < t < 1$. 

Jeffrey T. Neugebauer

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Fractional Calculus and Smallest Eigenvalues
Now

\[ Mu(t) = t^{\alpha-1} \left( \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds \right. \\
- t^{1-\alpha} \left. \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds \right). \]
Now

\[ Mu(t) = t^{\alpha - 1} \left( \int_0^1 \frac{(1 - s)^{\alpha - 1}}{\Gamma(\alpha)} p(s) u(s) ds \right. \]

\[ \left. - t^{1 - \alpha} \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} p(s) u(s) ds \right) . \]

Let

\[ v(t) = \int_0^1 \frac{(1 - s)^{\alpha - 1}}{\Gamma(\alpha)} p(s) u(s) ds - t^{1 - \alpha} \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} p(s) u(s) ds. \]
So \[ v(0) = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) u(s) ds > 0 \]
So $v(0) = \int_{0}^{1} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds > 0$ and

$$v'(1) = (\alpha - 1) \int_{0}^{1} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds - (\alpha - 1) \int_{0}^{1} \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} p(s)u(s)ds < 0.$$
So \( v(0) = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds > 0 \) and

\[
    v'(1) = (\alpha - 1) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds \\
    - (\alpha - 1) \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} p(s)u(s)ds < 0.
\]

So \( M : \mathcal{P} \setminus \{0\} \to \Omega \subset \mathcal{P}^o. \)
Notice that

\[ \Lambda u = Mu = \int_0^1 G(t, s)p(s)u(s)ds, \]
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if and only if
\[ u(t) = \frac{1}{\Lambda} \int_0^1 G(t, s)p(s)u(s)ds, \]
with \( u(0) = u(1) = 0 \). So the eigenvalues of (1),(3) are reciprocals of eigenvalues of \( M \), and conversely. Similarly, eigenvalues of (2),(3) are reciprocals of eigenvalues of \( N \), and conversely.
Notice that

\[ \Lambda u = Mu = \int_0^1 G(t, s)p(s)u(s)\,ds, \]

if and only if

\[ u(t) = \frac{1}{\Lambda} \int_0^1 G(t, s)p(s)u(s)\,ds, \]

if and only if

\[ D_{0+}^\alpha u(t) + \frac{1}{\Lambda} p(t)u(t) = 0, \quad 0 < t < 1, \]

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So the eigenvalues of (1),(3) are reciprocals of eigenvalues of \( M \), and conversely. Similarly, eigenvalues of (2),(3) are reciprocals of eigenvalues of \( N \), and conversely.
Theorem

Let $B$, $\mathcal{P}$, $M$, and $N$ be defined as earlier. Then $M$ (and $N$) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$. 
Theorem

Let $B$, $\mathcal{P}$, $M$, and $N$ be defined as earlier. Then $M$ (and $N$) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.

Theorem

Let $B$, $\mathcal{P}$, $M$, and $N$ be defined as earlier. Let $p(t) \leq q(t)$ on $[0,1]$. Let $\Lambda_1$ and $\Lambda_2$ be the eigenvalues defined in the previous theorem associated with $M$ and $N$, respectively, with the essentially unique eigenvectors $u_1$ and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(t) = q(t)$ on $[0,1]$. 

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Fractional Calculus and Smallest Eigenvalues
The following theorem is an immediate consequence of the relationship between the eigenvalues of $M$ and (1),(3), and the eigenvalues of $N$ and (2),(3), and the previous two theorems.
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**Theorem**

Assume the hypotheses of the previous theorem. Then there exists smallest positive eigenvalues $\lambda_1$ and $\lambda_2$ of (1),(3) and (2),(3), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to $\lambda_1$ and $\lambda_2$ may be chosen to belong to $P^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(t) = q(t)$ for all $t \in [0, 1]$. 