Disconjugacy and Differentiation for Solutions of Boundary Value Problems for Second Order Dynamic Equations on a Time Scale

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Many papers have been published working with derivatives and differences of solutions of boundary value problems with respect to the boundary data. A few to reference are Benchohra et al. [1], Ehrke et al. [3], Henderson [5], Hopkins et al. [6], and Lyons [7].
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However, they all stem from a theorem that Hartman [4] attributes to Peano dealing with solutions of initial value problems. Today, we will be adapting a theorem of Peano’s for initial value problems to a second order boundary value problem on a time scale.
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**Example**

A few examples that are not time scales are \( \mathbb{Q}, \mathbb{C}, (0, 1) \).
Jump Operators

Definition

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the **forward jump operator** $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the **backward jump operator** $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$
### Definition

If $\sigma(t) > t$, we say that $t$ is **right-scattered**, while if $\rho(t) < t$, we say that $t$ is **left-scattered**. Points that are both right- and left-scattered are called **isolated**.
Definition

If $\sigma(t) > t$, we say that $t$ is right-scattered, while if $\rho(t) < t$, we say that $t$ is left-scattered. Points that are both right- and left-scattered are called isolated.

Definition

If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then $t$ is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then $t$ is called left-dense. Points that are both right- and left-dense are called dense.
Scattered and Dense

**Definition**
If \( \sigma(t) > t \), we say that \( t \) is **right-scattered**, while if \( \rho(t) < t \), we say that \( t \) is **left-scattered**. Points that are both right- and left-scattered are called **isolated**.

**Definition**
If \( t < \sup T \) and \( \sigma(t) = t \), then \( t \) is called **right-dense**, and if \( t > \inf T \) and \( \rho(t) = t \), then \( t \) is called **left-dense**. Points that are both right- and left-dense are called **dense**.

**Definition**
The **graininess function** \( \mu : T \to [0, \infty) \) is defined by

\[
\mu(t) := \sigma(t) - t.
\]
Table 1.1. Classification of Points

<table>
<thead>
<tr>
<th>Type</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$ right-scattered</td>
<td>$t &lt; \sigma(t)$</td>
</tr>
<tr>
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</tr>
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</tr>
<tr>
<td>$t$ left-dense</td>
<td>$\rho(t) = t$</td>
</tr>
<tr>
<td>$t$ isolated</td>
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</tr>
<tr>
<td>$t$ dense</td>
<td>$\rho(t) = t = \sigma(t)$</td>
</tr>
</tbody>
</table>

Figure 1.1. Classifications of Points

- $t_1$ is left-dense and right-dense
- $t_2$ is left-dense and right-scattered
- $t_3$ is left-scattered and right-dense
- $t_4$ is left-scattered and right-scattered

($t_1$ is dense and $t_4$ is isolated)
Example

(1) If $T = \mathbb{R}$, then we have for any $t \in \mathbb{R}$

$$\sigma(t) = \inf s \in \mathbb{R} : s > t = \inf(t, \infty) = t$$

and similarly, $\rho(t) = t$. Hence, every point $t \in \mathbb{R}$ is dense. The graininess function $\mu$ turns out to be

$$\mu(t) \equiv 0 \text{ for all } t \in T.$$
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(2) If $T = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$

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and similarly, $\rho(t) = t - 1$. Hence, every point $t \in \mathbb{Z}$ is isolated. The graininess function $\mu$ in this case is

$$\mu(t) \equiv 1 \text{ for all } t \in T.$$
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(3) If $T = h\mathbb{Z} := hk : k \in \mathbb{Z}$, then we have for any $t \in h\mathbb{Z}$

$$\sigma(t) = \inf s \in \mathbb{Z} : s > t = \inf t + h, t + 2ht + 3h, \ldots = t + h$$

and similarly, $\rho(t) = t - h$. Hence, every point $t \in \mathbb{Z}$ is isolated. The graininess function $\mu$ in this case is

$$\mu(t) \equiv h \text{ for all } t \in T.$$
Delta Derivative

**Definition**

Define the set

\[ T^\kappa = \begin{cases} \mathbb{T} \setminus \{\rho(\sup T), \sup T\} & \text{if } \sup T < \infty \\ \mathbb{T} & \text{if } \sup T = \infty \end{cases} \]
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Definition

Assume \( f : T \rightarrow \mathbb{R} \) is a function and let \( t \in T^{\kappa} \). Then we define \( f^\Delta(t) \) to be the number (provided it exists) with the property that given any \( \varepsilon > 0 \), there is a neighborhood \( U \) of \( t \) such that

\[ |f(\sigma(t)) - f(s) - f^\Delta(t) [\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U.\]
Delta Derivative

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|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon |\sigma(t) - s| \quad \text{for all } s \in U.
\]

We call \( f^\Delta(t) \) the \textbf{delta (Hilger) derivative} of \( f \) at \( t \).
Theorem

Assume \( f : \mathbb{T} \to \mathbb{R} \) is a function and let \( t \in \mathbb{T}^\kappa \)appa. Then we have the following:

(i) If \( f \) is differentiable at \( t \), then \( f \) is continuous at \( t \).
**Theorem**

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$appa. Then we have the following:

(i) If $f$ is differentiable at $t$, then $f$ is continuous at $t$.

(ii) If $f$ is continuous at $t$ and $t$ is right-scattered, then $f$ is differentiable at $t$ with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$
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(iii) If $t$ is right-dense, then $f$ is differentiable at $t$ if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s} =: f^{\Delta}(t)$$

exists as a finite number.
Properties of the Delta Derivatives

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(iv) If $f$ is differentiable at $t$, then

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Properties of the Delta Derivatives

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Corollary

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(iii) Suppose the same conditions as above and let $\mathbb{T} = h\mathbb{Z}$. Then $f^\Delta(t) = \frac{f(t + h) - f(t)}{h}$.
Corollary

(i) **Suppose the same conditions as above and let** \( T = \mathbb{R} \). Then \( f^\Delta(t) = f'(t) \).

(ii) **Suppose the same conditions as above and let** \( T = \mathbb{Z} \). Then \( f^\Delta(t) = \Delta f(t) = f(t + 1) - f(t) \).

(iii) **Suppose the same conditions as above and let** \( T = h\mathbb{Z} \). Then
\[
 f^\Delta(t) = \frac{f(t + h) - f(t)}{h} .
\]
Consider the second order boundary value problem

\[ y^\Delta \Delta = f(t, y, y^\Delta), \quad t \in h\mathbb{Z} \]  \hspace{1cm} (1)

satisfying

\[ y(t_1) = y_1, \quad y(t_2) = y_2 \]  \hspace{1cm} (2)

where \( t_1 + h < t_2 \) in \( h\mathbb{Z} \) and \( y_1, y_2 \in \mathbb{R} \).
The Boundary Value Problem

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where \( t_1 + h < t_2 \) in \( h\mathbb{Z} \) and \( y_1, y_2 \in \mathbb{R} \).

**Definition**

The *variational equation* along a solution \( y(t) \) of (1) is given by

\[ z^\Delta\Delta = \frac{\partial f}{\partial d_1}(t, y, y^\Delta)z + \frac{\partial f}{\partial d_2}(t, y, y^\Delta)z^\Delta. \]  \hspace{1cm} (3)
Here are some assumptions we will place upon (1):

(i) $f(t, d_1, d_2) : h\mathbb{Z} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous,
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(iii) the equation $d_3 = f(t, d_1, d_2)$ can be solved for $d_1$ as a continuous function of $d_2$ and $d_3$, for each $t \in h\mathbb{Z}$.
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Remark

Given a solution $y(t)$ of (1), condition (iii) yields that $y(t)$ exists on all of $h\mathbb{Z}$. 
Recall that we are seeking an analogue of a theorem of Peano’s for boundary value problems. Therefore, we must first look at the work in the context of the initial value problem (1) satisfying

\[ y(t_0) = c_1, \quad y^\Delta(t_0) = c_2, \quad t_0 \in h\mathbb{Z}, \quad c_1, c_2 \in \mathbb{R}. \] (4)
Recall that we are seeking an analogue of a theorem of Peano’s for boundary value problems. Therefore, we must first look at the work in the context of the initial value problem (1) satisfying

$$y(t_0) = c_1, \quad y^A(t_0) = c_2, \quad t_0 \in h\mathbb{Z}, \quad c_1, c_2 \in \mathbb{R}. \quad (4)$$

Condition (iii) implies that solutions of (1), (4) are unique on all of $h\mathbb{Z}$. We denote the unique solution of (1), (2) by $u(t, t_0, c_1, c_2)$. 
Theorem (Continuous Dependence wrt Initial Values)

Assume conditions (i) and (iii) hold. Let \( t_0 \in h\mathbb{Z} \) and \( c_1, c_2 \in \mathbb{R} \) be given. Then for all \( \epsilon > 0 \) and for all \( k \in \mathbb{N} \), there exists \( \delta(\epsilon, k, t_0, c_1, c_2) > 0 \) such that \( |c_1 - e_1| < \delta \) and \( |c_2 - e_2| < \delta \) imply \( |u(t, t_0, c_1, c_2) - u(t, t_0, e_1, e_2)| < \epsilon \) for \( t \in [t_0 - hk, t_0 + hk] \) and \( e_1, e_2 \in \mathbb{R} \).
Preliminary Theorems for IVPs

Theorem (Continued)

where

\[
A_1(t) = \int_0^1 \frac{\partial f}{\partial d_1}(t, su(t, t_0 + h, c_1, c_2)) \\
+ (1 - s)u(t, t_0, c_1, c_2), u^\Delta(t, t_0, c_1, c_2))ds,
\]

\[
A_2(t) = \int_0^1 \frac{\partial f}{\partial d_1}(t, u(t, t_0 + h, c_1, c_2), su^\Delta(t, t_0 + h, c_1, c_2)) \\
+ (1 - s)u^\Delta(t, t_0, c_1, c_2))ds,
\]
Definition (Generalized Zero)

Let $\nu : h\mathbb{Z} \rightarrow \mathbb{R}$. We say $\nu$ has a \textit{generalized zero} at $t_0 \in h\mathbb{Z}$ provided either

1. $\nu(t_0) = 0$ or

2. There exists $k \in \mathbb{N}$ such that 
   $(-1)^k \nu(t_0 - hk) \nu(t_0) > 0$ and if $k > 1$,
   then $\nu(t_0 - hk + h) = \cdots = \nu(t_0 - h)$ = 0.
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The following two conditions are disconjugate type conditions for (1) and provide uniqueness for the solutions of (1), (2):

**Definition**

The nonlinear dynamic equation (1) is said to satisfy Property U on \( h\mathbb{Z} \) if whenever \( y_1(t) \) and \( y_2(t) \) are solutions of (1) such that \( y_1(t) - y_2(t) \) has a generalized zero at \( t_1 \) and \( t_2 \) with \( t_1 + h < t_2 \in h\mathbb{Z} \), then \( y_1(t) - y_2(t) \equiv 0 \) on \( h\mathbb{Z} \).
Preliminary Theorems for Boundary Values

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**Definition**

The linear dynamic equation

$$s^{\Delta\Delta} = M(t)s(t) + N(t)s^{\Delta} \quad (5)$$

is said to satisfy **Property U** on $h\mathbb{Z}$ provided there is no nontrivial solution $s(t)$ of (5) such that $s(t)$ has a generalized zero at $t_1$ and $t_2$ with $t_1 + h < t_2 \in h\mathbb{Z}$. 

Theorem (Continuous Dependence wrt Boundary Values)

Assume conditions (i) and (iii) hold and that (1) satisfies Property U on $h\mathbb{Z}$. Let $y(t)$ be a solution of (1). Also, let $t_1 + h < t_2$ in $h\mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$ be given. Then there exists $\epsilon > 0$ such that if $\delta_1, \delta_2 \in \mathbb{R}$ with $|\delta_i| < \epsilon$ for $i = 1, 2$, the boundary value problem for (1) satisfying

$$w(t_1) = y(t_1) + \delta_1, \ w(t_2) = y(t_2) + \delta_2$$

has a unique solution

$$w(t, t_1, t_2, y(t_1) + \delta_1, y(t_2) + \delta_2).$$
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Moreover, as $\epsilon \rightarrow 0$, this solution converges to $y(t)$ on $h\mathbb{Z}$. 

We now provide and prove two results which are analogues to Peano's theorems for initial value problems.
Theorem (Continuous Dependence wrt Boundary Values)

Assume conditions (i) and (iii) hold and that (1) satisfies Property U on $h\mathbb{Z}$. Let $y(t)$ be a solution of (1). Also, let $t_1 + h < t_2$ in $h\mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$ be given. Then there exists $\epsilon > 0$ such that if $\delta_1, \delta_2 \in \mathbb{R}$ with $|\delta_i| < \epsilon$ for $i = 1, 2$, the boundary value problem for (1) satisfying

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We now provide and prove two results which are analogues to Peano’s theorems for initial value problems.
Theorem

Assume conditions (i)-(iii) are satisfied, that (1) satisfies Property U on $h\mathbb{Z}$, and that (3) satisfies Property U along all solutions of (1). Suppose $y(t, t_1, t_2, y_1, y_2)$ is a solution of (1) on $h\mathbb{Z}$ where $t_1 + h < t_2$ in $h\mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$.
Assume conditions (i)-(iii) are satisfied, that (1) satisfies Property U on $h\mathbb{Z}$, and that (3) satisfies Property U along all solutions of (1). Suppose $y(t, t_1, t_2, y_1, y_2)$ is a solution of (1) on $h\mathbb{Z}$ where $t_1 + h < t_2$ in $h\mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$. Then for $i = 1, 2$, $z_i(t) := \frac{\partial y}{\partial y_i}$ exists on $h\mathbb{Z}$ and is a solution of (3) along $y(t)$ that satisfies

$$z_1(t_1) = 1, \quad z_1(t_2) = 0$$

$$z_2(t_1) = 0, \quad z_2(t_2) = 1.$$
We only look at \( z_1(t) \) as \( z_2(t) \) is essentially the same proof. Let \( \epsilon > 0 \) as in the Continuous Dependence Theorem, and let \( 0 \leq |h| < \epsilon \) be given. Consider

\[
z_{1p}(t) = \frac{1}{p} [y(t, y_1 + p) - y(t, y_1)].
\]
We only look at $z_1(t)$ as $z_2(t)$ is essentially the same proof. Let $\epsilon > 0$ as in the Continuous Dependence Theorem, and let $0 \leq |h| < \epsilon$ be given. Consider

$$z_{1p}(t) = \frac{1}{p} [y(t, y_1 + p) - y(t, y_1)].$$

We want to show $\lim_{p \to 0} z_{1p}(t)$ exists on $h\mathbb{Z}$. 
We only look at $z_1(t)$ as $z_2(t)$ is essentially the same proof. Let $\epsilon > 0$ as in the Continuous Dependence Theorem, and let $0 \leq |h| < \epsilon$ be given. Consider

\[ z_{1p}(t) = \frac{1}{p} [y(t, y_1 + p) - y(t, y_1)]. \]

We want to show $\lim_{p \to 0} z_{1p}(t)$ exists on $h\mathbb{Z}$. For $h \neq 0$,

\[ z_{1p}(t_1) = \frac{1}{p} [y_1 + p - y_1] = 1 \]

and

\[ z_{1p}(t_2) = \frac{1}{p} [y_2 - y_2] = 0. \]
We only look at $z_1(t)$ as $z_2(t)$ is essentially the same proof. Let $\epsilon > 0$ as in the Continuous Dependence Theorem, and let $0 \leq |h| < \epsilon$ be given. Consider

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$$z_{1p}(t_2) = \frac{1}{p} [y_2 - y_2] = 0.$$

View $z_{1p}(t)$ in terms of a solution of an IVP at $t_1$.

Let $\sigma_2 = y^\Delta(t_1, y_1)$ and $\epsilon_2 = y^\Delta(t_1, y_1 + p) - \sigma_2$.
By Continuous Dependence, $\epsilon_2 \to 0$ as $p \to 0$. 
Now,

\[ z_1 p(t) = \frac{1}{p} [u(t, t_1, y_1 + p, \sigma_2 + \epsilon_2) - u(t, t_1, y_1, \sigma_2)] \]

\[ = \frac{1}{p} [u(t, t_1, y_1 + p, \sigma_2 + \epsilon_2) - u(t, t_1, y_1, \sigma_2 + \epsilon_2) + u(t, t_1, y_1, \sigma_2 + \epsilon_2) - u(t, t_1, y_1, \sigma_2)] \]

\[ = \frac{1}{p} \left[ \beta_1(t, u(\cdot, t_1, y_1 + \bar{p}, \sigma_2 + \epsilon_2)) p \right. \]

\[ + \beta_2(t, u(\cdot, t_1, y_1, \sigma_2 + \bar{\epsilon}_2)) \epsilon_2 \]

where \( \beta_i(t, u(\cdot)) \), \( i = 1, 2 \), denotes solutions of (3) along \( u(\cdot) \) satisfying

\[ \beta_1(t_1, u(\cdot)) = 1 \quad \beta_2(t_1, u(\cdot)) = 0 \]

\[ \beta_1^\Delta(t_1, u(\cdot)) = 0 \quad \beta_2^\Delta(t_1, u(\cdot)) = 1. \]

In addition, \( \bar{p} \) is between 0 and \( h \) and \( \bar{\epsilon}_2 \) is between 0 and \( \epsilon_2 \).
Thus, \( z_{1p}(t) = \beta_1(t, u(\cdot, t_1, y_1 + \bar{p}, \sigma_2 + \epsilon_2)) + \frac{\epsilon_2}{p} \beta_2(t, u(\cdot, t_1, y_1, \sigma_2 + \bar{\epsilon}_2)). \)
Thus, $z_{1p}(t) = \beta_1(t, u(\cdot, t_1, y_1 + p, \sigma_2 + \epsilon_2)) + \frac{\epsilon_2}{p} \beta_2(t, u(\cdot, t_1, y_1, \sigma_2 + \bar{\epsilon}_2))$. We need to show $\lim_{p \to 0} \frac{\epsilon_2}{p}$ exists.
Thus, \( z_1 p(t) = \beta_1(t, u(\cdot, t_1, y_1 + \bar{p}, \sigma_2 + \epsilon_2)) + \frac{\epsilon_2}{p} \beta_2(t, u(\cdot, t_1, y_1, \sigma_2 + \bar{\epsilon}_2)). \)

We need to show \( \lim_{p \to 0} \frac{\epsilon_2}{p} \) exists.

Recall, \( z_1 p(t_2) = 0 \), and hence

\[
\frac{\epsilon_2}{p} = -\frac{\beta_1(t_2, u(\cdot, t_1, y_1 + \bar{p}, \sigma_2 + \epsilon_2))}{\beta_2(t_2, u(\cdot, t_1, y_1, \sigma_2 + \bar{\epsilon}_2))}
\]

which has nonzero denominator by Property U for the variational equation.
Let \( L := \lim_{p \to 0} \frac{\varepsilon_2}{p} \).
Let \( L := \lim_{p \to 0} \frac{\epsilon_2}{p} \).

Then,

\[
z_1(t) = \lim_{p \to 0} z_{1p}(t)
= \lim_{p \to 0} \left[ \beta_1(t, u(\cdot, t_1, y_1 + \bar{p}, \sigma_2 + \epsilon_2)) + \frac{\epsilon_2}{p} \beta_2(t, u(\cdot, t_1, y_1, \sigma_2 + \bar{\epsilon}_2)) \right]
= \beta_1(t, u(\cdot)) + L \beta_2(t, u(\cdot))
\]

which solves the variational equation along \( u(\cdot) \).
Let \( L := \lim_{p \to 0} \frac{\epsilon_2}{p} \).

Then,

\[
z_1(t) = \lim_{p \to 0} z_{1p}(t)
= \lim_{p \to 0} \left[ \beta_1(t, u(\cdot, t_1, y_1 + \bar{p}, \sigma_2 + \epsilon_2))
+ \frac{\epsilon_2}{p} \beta_2(t, u(\cdot, t_1, y_1, \sigma_2 + \overline{\epsilon_2})) \right]
= \beta_1(t, u(\cdot)) + L \beta_2(t, u(\cdot))
\]

which solves the variational equation along \( u(\cdot) \).

Also, \( z_1(t_1) = \lim_{p \to 0} z_{1p}(t_1) = 1 \) and \( z_1(t_2) = \lim_{p \to 0} (t_2) = 0 \).
Let \( L := \lim_{p \to 0} \frac{\epsilon_2}{p} \).

Then,

\[
z_1(t) = \lim_{p \to 0} z_{1p}(t) = \lim_{p \to 0} \left[ \beta_1(t, u(\cdot, t_1, y_1 + p, \sigma_2 + \epsilon_2)) + \frac{\epsilon_2}{p} \beta_2(t, u(\cdot, t_1, y_1, \sigma_2 + \epsilon_2)) \right] = \beta_1(t, u(\cdot)) + L \beta_2(t, u(\cdot))
\]

which solves the variational equation along \( u(\cdot) \).

Also, \( z_1(t_1) = \lim_{p \to 0} z_{1p}(t_1) = 1 \) and \( z_1(t_2) = \lim_{p \to 0} (t_2) = 0 \).

Therefore, \( z_1(t) = \frac{\partial y}{\partial y_1}(t) \).
Assume conditions (i)-(iii) hold and that (1) satisfies Property U on $h\mathbb{Z}$. Let $y(t, t_1, t_2, y_1, y_2)$ be a solution of (1), (2) on $h\mathbb{Z}$ where $t_1 + h < t_2$ in $h\mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$. 
Boundary Values Result

**Theorem**

Assume conditions (i)-(iii) hold and that (1) satisfies Property U on $h\mathbb{Z}$. Let $y(t, t_1, t_2, y_1, y_2)$ be a solution of (1), (2) on $h\mathbb{Z}$ where $t_1 + h < t_2$ in $h\mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$. Then for $i = 1, 2$, $\nu_i(t) := y^{\Delta_t}(t, t_1, t_2, y_1, y_2)$ is a solution of the linear dynamic equation

\[
\nu_i^{\Delta\Delta} = A_{1i}(t)\nu_i(t) + A_{2i}(t)\nu_i^\Delta
\]
Theorem

Assume conditions (i)-(iii) hold and that (1) satisfies Property U on $h\mathbb{Z}$. Let $y(t, t_1, t_2, y_1, y_2)$ be a solution of (1), (2) on $h\mathbb{Z}$ where $t_1 + h < t_2$ in $h\mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$. Then for $i = 1, 2$, $\nu_i(t) := y^{\Delta t_i}(t, t_1, t_2, y_1, y_2)$ is a solution of the linear dynamic equation

$$
\nu^\Delta_{\Delta} = A_{1i}(t)\nu_i(t) + A_{2i}(t)\nu^\Delta_i
$$

where

$$
A_{1i}(t) = \int_0^1 \frac{\partial f}{\partial d_1}(t, sy(t, t_i + h) + (1 - s)y(t, t_i), y^\Delta(t, t_i + h))ds,
$$

$$
A_{2i}(t) = \int_0^1 \frac{\partial f}{\partial d_2}(t, y(t, t_i), sy^\Delta(t, t_i + h) + (1 - s)y^\Delta(t, t_i))ds,
$$
Boundary Values Result

**Theorem**

Assume conditions (i)-(iii) hold and that (1) satisfies Property U on $h\mathbb{Z}$. Let $y(t, t_1, t_2, y_1, y_2)$ be a solution of (1), (2) on $h\mathbb{Z}$ where $t_1 + h < t_2$ in $h\mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$. Then for $i = 1, 2$, $\nu_i(t) := y^\Delta_i(t, t_1, t_2, y_1, y_2)$ is a solution of the linear dynamic equation

$$
\nu_i^{\Delta\Delta} = A_1i(t)\nu_i(t) + A_2i(t)\nu_i^\Delta
$$

where

$$
A_1i(t) = \int_0^1 \frac{\partial f}{\partial d_1}(t, sy(t, ti + h) + (1-s)y(t, ti), y^\Delta(t, ti + h))ds,
$$

$$
A_2i(t) = \int_0^1 \frac{\partial f}{\partial d_2}(t, y(t, ti), sy^\Delta(t, ti + h) + (1-s)y^\Delta(t, ti))ds,
$$

with boundary conditions

$$
\nu_1(t_1) = -y^\Delta(t_1, t_1 + h, t_2, y_1, y_2), \quad \nu_1(t_2) = 0,
$$

$$
\nu_2(t_1) = 0, \quad \nu_2(t_2) = -y^\Delta(t_2, t_1, t_2 + h, y_1, y_2).
$$
We only look at $\nu_1(t)$ as $\nu_2(t)$ is similar.

First,

$$
\nu_1^{\Delta\Delta}(t) = \left[y^{\Delta t_1}(t, t_1, t_2, y_1, y_2)\right]^{\Delta\Delta}
= \frac{1}{h}[y^{\Delta\Delta}(t, t_1 + h) - y^{\Delta\Delta}(t, t_1)]
= \frac{1}{h}[f(t, y(t, t_1 + h), y^{\Delta}(t, t_1 + h)) - f(t, y(t, t_1), y^{\Delta}(t, t_1 + h))
+ f(t, y(t, t_1), y^{\Delta}(t, t_1 + h) - f(t, y(t, t_1), y^{\Delta}(t, t_1))]
= \int_0^1 \frac{\partial f}{\partial d_1}(t, sy(t, t_1 + h) + (1 - s)y(t, t_1), y^{\Delta}(t, t_1 + h))ds
\times \frac{y(t, t_1 + h) - y(t, t_1)}{h}
+ \int_0^1 \frac{\partial f}{\partial d_2}(t, y(t, t_1), sy^{\Delta}(t, t_1 + h) + (1 - s)y^{\Delta}(t, t_1))ds
\times \frac{y^{\Delta}(t, t_1 + h) - y^{\Delta}(t, t_1)}{h}
= A_{11}\nu_1(t) + A_{12}\nu_1^{\Delta}(t).
$$
In addition,

\[ \nu_1(t_1) = y^\Delta_{t_1}(t_1, t_1, t_2, y_1, y_2) \]

\[ = \frac{1}{h} [ y(t_1, t_1 + h, t_2, y_1, y_2) - y(t_1, t_1, t_2, y_1, y_2) ] \]

\[ = \frac{1}{h} [ y(t_1, t_1 + h) - y(t_1 + h, t_1 + h) \]

\[ + y(t_1 + h, t_1 + h) - y_1 ] \]

\[ = - y^\Delta(t_1, t_1 + h) + \frac{1}{h} [ y_1 - y_1 ] \]

\[ = - y^\Delta(t_1, t_1 + h). \]
In addition,

\[ \nu_1(t_1) = y^{A_{t_1}}(t_1, t_1, t_2, y_1, y_2) \]

\[ = \frac{1}{h} [y(t_1, t_1 + h, t_2, y_1, y_2) - y(t_1, t_1, t_2, y_1, y_2)] \]

\[ = \frac{1}{h} [y(t_1, t_1 + h) - y(t_1 + h, t_1 + h) \]

\[ + y(t_1 + h, t_1 + h) - y_1] \]

\[ = -y^{A}(t_1, t_1 + h) + \frac{1}{h} [y_1 - y_1] \]

\[ = -y^{A}(t_1, t_1 + h). \]

and

\[ \nu_1(t_2) = y^{A_{t_1}}(t_2, t_1, t_2, y_1, y_2) \]

\[ = \frac{1}{h} [y(t_2, t_1 + h, t_2, y_1, y_2) - y(t_2, t_1, t_2, y_1, y_2)] \]

\[ = \frac{1}{h} [y_2 - y_2] = 0. \]
Corollary to Difference Equations

Notice that if we let $h = 1$ in the above work and theorems, we obtain the following difference equation over $h\mathbb{Z} = \mathbb{Z}$

$$\Delta\Delta y = f(t, y, \Delta y) \ [y(t + 2) = f(t, y, y(t + 1))] \quad (6)$$

satisfying

$$y(t_1) = y_1, \ y(t_2) = y_2 \quad (7)$$

where $t_1 + 1 < t_2$ in $\mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$. 
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Notice that if we let $h = 1$ in the above work and theorems, we obtain the following difference equation over $h\mathbb{Z} = \mathbb{Z}$

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where $t_1 + 1 < t_2$ in $\mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$.

Also note, the variational equation can be written

$$z(t + 2) = \frac{\partial f}{\partial d_1} z(t) + \frac{\partial f}{\partial d_2} z(t + 1). \quad (8)$$
The previous results immediately lead to the following corollaries.
The previous results immediately lead to the following corollaries.

**Corollary**

Assume conditions (i)-(iii) are satisfied, that (1) satisfies Property U on \( h\mathbb{Z} = \mathbb{Z} \), and that the variational equation (8) satisfies Property U along all solutions of (1). Let \( y(t, t_1, t_2, y_1, y_2) \) be a solution of (1), (2) on \( \mathbb{Z} \) where \( t_1 + 1 < t_2 \) in \( \mathbb{Z} \) and \( y_1, y_2 \in \mathbb{R} \).
Corollary to Difference Equations

The previous results immediately lead to the following corollaries.

**Corollary**

Assume conditions (i)-(iii) are satisfied, that (1) satisfies Property U on $h\mathbb{Z} = \mathbb{Z}$, and that the variational equation (8) satisfies Property U along all solutions of (1). Let $y(t, t_1, t_2, y_1, y_2)$ be a solution of (1), (2) on $\mathbb{Z}$ where $t_1 + 1 < t_2$ in $\mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$. Then for $i = 1, 2$, $z_i(t) := \frac{\partial y}{\partial t_i}$ exists and solves (8) along $y(t)$ satisfying

\[
z_1(t_1) = 1, \quad z_1(t_2) = 0
\]

\[
z_2(t_1) = 0, \quad z_2(t_2) = 1.
\]
Corollary to Difference Equations

Corollary

Assume conditions (i)-(iii) are satisfied and that (1) satisfies Property U on $\mathbb{Z}$. Let $y(t, t_1, t_2, y_1, y_2)$ be a solution of (1), (2) on $\mathbb{Z}$ where $t_1 + 1 < t_2$ in $\mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$. Then for $i = 1, 2$, $\nu_i(t) := \Delta_{t_1} y(t, t_1, t_2, y_1, y_2)$ solves $\nu_i(t + 2) = A_{i1}(t) \nu_i(t) + A_{i2}(t) \nu_i(t + 1)$ with $\nu_i(t_j) = -\Delta_t y(t, t_j + 1) |_{t = t_1} \delta_{ij}$, for $j = 1, 2$, where $A_{i1} = \int_1^0 \frac{\partial f}{\partial d_1}(t, \cdot) dt + (1 - s) y(t, t_1)$, $A_{i2} = \int_1^0 \frac{\partial f}{\partial d_2}(t, \cdot) dt + (1 - s) y(t + 1, t_1)$.

These are very similar results to those found in Benchohra et al. [1] and Hopkins et al. [4].
Corollary to Difference Equations

**Corollary**

Assume conditions (i)-(iii) are satisfied and that (1) satisfies Property $U$ on $\mathbb{Z}$. Let $y(t, t_1, t_2, y_1, y_2)$ be a solution of (1), (2) on $\mathbb{Z}$ where $t_1 + 1 < t_2$ in $\mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$. Then for $i = 1, 2$, $\nu_i(t) := \Delta_{m_1} y(t, t_1, t_2, y_1, y_2)$ solves

$$
\nu_i(t + 2) = A_{i1}(t)\nu_i(t) + A_{i2}\nu_i(t + 1)
$$

$$
\nu_i(t_j) = -\Delta_t y(t, t_j + 1)|_{t=t_1} \delta_{ij}, \text{ for } j = 1, 2,
$$

These are very similar results to those found in Benchohra et al. [1] and Hopkins et al. [4].
Corollary to Difference Equations

**Corollary**

Assume conditions (i)-(iii) are satisfied and that (1) satisfies Property U on \( \mathbb{Z} \). Let \( y(t, t_1, t_2, y_1, y_2) \) be a solution of (1), (2) on \( \mathbb{Z} \) where \( t_1 + 1 < t_2 \) in \( \mathbb{Z} \) and \( y_1, y_2 \in \mathbb{R} \). Then for \( i = 1, 2 \), \( \nu_i(t) := \Delta_{m_1}y(t, t_1, t_2, y_1, y_2) \) solves

\[
\nu_i(t + 2) = A_{i1}(t)\nu_i(t) + A_{i2}\nu_i(t + 1)
\]

\[
\nu_i(t_j) = -\Delta_t y(t, t_j + 1)|_{t=t_1}\delta_{ij}, \text{ for } j = 1, 2,
\]

where

\[
A_{i1} = \int_0^1 \frac{\partial f}{\partial d_1}(t, sy(t, t_i + 1) + (1 - s)y(t, t_i), y(t + 1, t_i + 1))ds,
\]

\[
A_{i2} = \int_0^1 \frac{\partial f}{\partial d_2}(t, y(t, t_i), sy(t + 1, t_i + 1) + (1 - s)y(t + 1, t_i))ds.
\]
Corollary to Difference Equations

Assume conditions (i)-(iii) are satisfied and that (1) satisfies Property U on \( \mathbb{Z} \). Let \( y(t, t_1, t_2, y_1, y_2) \) be a solution of (1), (2) on \( \mathbb{Z} \) where \( t_1 + 1 < t_2 \) in \( \mathbb{Z} \) and \( y_1, y_2 \in \mathbb{R} \). Then for \( i = 1, 2 \), \( \nu_i(t) := \Delta m_1 y(t, t_1, t_2, y_1, y_2) \) solves

\[
\nu_i(t + 2) = A_{i1}(t)\nu_i(t) + A_{i2}\nu_i(t + 1)
\]

\[
\nu_i(t_j) = -\Delta t y(t, t_j + 1)|_{t=t_1}\delta_{ij}, \text{ for } j = 1, 2,
\]

where

\[
A_{i1} = \int_0^1 \frac{\partial f}{\partial d_1}(t, sy(t, t_i + 1) + (1 - s)y(t, t_i), y(t + 1, t_i + 1))ds,
\]

\[
A_{i2} = \int_0^1 \frac{\partial f}{\partial d_2}(t, y(t, t_i), sy(t + 1, t_i + 1) + (1 - s)y(t + 1, t_i))ds.
\]

These are very similar results to those found in Benchohra et al. [1] and Hopkins et al. [4].
Next, if we let \( h \to 0 \) in the main results, we obtain the following differential equation because of conditions (i)-(iii)

\[ y'' = f(t, y, y') \]  (9)

satisfying

\[ y(t_1) = y_1, \ y(t_2) = y_2 \]  (10)

where \( t_1 < t_2 \) in \( \mathbb{R} \) and \( y_1, y_2 \in \mathbb{R} \).
Next, if we let $h \to 0$ in the main results, we obtain the following differential equation because of conditions (i)-(iii)

$$y'' = f(t, y, y')$$  \hspace{1cm} (9)

satisfying

$$y(t_1) = y_1, \ y(t_2) = y_2$$  \hspace{1cm} (10)

where $t_1 < t_2$ in $\mathbb{R}$ and $y_1, y_2 \in \mathbb{R}$.

Also, the variational equation becomes

$$z'' = \frac{\partial f}{\partial d_1} z + \frac{\partial f}{\partial d_2} z'.$$  \hspace{1cm} (11)
Assume conditions (i)-(iii) are satisfied, that (9) satisfies property U on $\mathbb{R}$ (disconjugacy), and that the variational equation (11) satisfies Property U along all solutions of (9) (disconjugacy). Let $y(t, t_1, t_2, y_1, y_2)$ be a solution of (9), (10) on $\mathbb{R}$ where $t_1 < t_2, y_1, y_2 \in \mathbb{R}$.
Corollary to Differential Equations

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Assume conditions (i)-(iii) are satisfied, that (9) satisfies property U on \( \mathbb{R} \) (disconjugacy), and that the variational equation (11) satisfies Property U along all solutions of (9) (disconjugacy). Let \( y(t, t_1, t_2, y_1, y_2) \) be a solution of (9), (10) on \( \mathbb{R} \) where \( t_1 < t_2, y_1, y_2 \in \mathbb{R} \). Then

(a) for \( i = 1, 2 \), \( z_i := \frac{\partial f}{\partial t_i} \) exists and solves (11) along \( y(t) \) satisfying

\[ z_i(t_j) = \delta_{ij}, \quad \text{for } j = 1, 2. \]
Corollary to Differential Equations

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Assume conditions (i)-(iii) are satisfied, that (9) satisfies property U on \( \mathbb{R} \) (disconjugacy), and that the variational equation (11) satisfies Property U along all solutions of (9) (disconjugacy). Let \( y(t, t_1, t_2, y_1, y_2) \) be a solution of (9), (10) on \( \mathbb{R} \) where \( t_1 < t_2, y_1, y_2 \in \mathbb{R} \). Then

(a) for \( i = 1, 2, \ z_i := \frac{\partial f}{\partial t_i} \) exists and solves (11) along \( y(t) \) satisfying

\[
z_i(t_j) = \delta_{ij}, \quad \text{for} \ j = 1, 2.
\]

(b) for \( i = 1, 2, \ \nu_i := \frac{\partial f}{\partial y_i} \) exists and solves (11) along \( y(t) \) satisfying

\[
\nu_i(t_j) = -y'(t_j)\delta_{ij}.
\]

This result is similar to that found in Ehrke et al. [2].
**Corollary to Differential Equations**

**Corollary**

Assume conditions (i)-(iii) are satisfied, that (9) satisfies property U on $\mathbb{R}$ (disconjugacy), and that the variational equation (11) satisfies Property U along all solutions of (9) (disconjugacy). Let $y(t, t_1, t_2, y_1, y_2)$ be a solution of (9), (10) on $\mathbb{R}$ where $t_1 < t_2, y_1, y_2 \in \mathbb{R}$. Then

(a) for $i = 1, 2$, $z_i := \frac{\partial f}{\partial t_i}$ exists and solves (11) along $y(t)$ satisfying

$$z_i(t_j) = \delta_{ij}, \text{ for } j = 1, 2.$$

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THANK YOU!