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Proof, Proving and Mathematics Curriculum

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Abstract

The National Council of Teachers of Mathematics, a US based teachers association, strongly encourages teachers to make proof and reasoning an integral part of student mathematics. However, the literature shows that, far from being integral, proving remains compartmentalized within the North American school curriculum and is restricted to a specific mathematical domain. Consequently, students suffer in their understandings and are ill prepared for the rigorous mathematical proving that many of them will encounter later at the postsecondary level. The literature suggests that compartmentalization is mainly due to teacher's lack of experience in proof and proving and their subsequent inability to guide students through the various stages of mathematical justifications. In this article, a brief overview of the history of proof is provided. Also, the nature of proof is explained and its importance in school mathematics is argued for. Teachers can help students better if they are aware of the stages that students experience over time as they become progressively more sophisticated at proof and proving tasks. By using a developmental model of proving, teachers are more likely to guide students effectively as they move from one stage to the other. Balacheff (1988) provides one of the most commonly used hierarchies to categorize student proof schemes. Few articles within the literature, however, apply this taxonomy in explicit ways by aligning examples of possible efforts to solve proof tasks with Balacheff's stages of proof. This article illustrates in accordance with Balacheff's taxonomy and by using examples, how a student might tackle a proof task. It is also argued that even though explanatory proofs are relevant to school mathematics, overemphasis on verbal proofs may result in watered down mathematical proofs. With an example I also demonstrate that students can transition fairly easily from explanatory proof to formal two –column proof.

Key Words: Mathematical proof, school mathematics, Naïve Empiricism, Crucial Experiment, Generic Example, Thought Experiment, Explanatory Proof, Two- column proof

Introduction

Proof is fundamental to mathematics. Mathematical proof has been regarded as one of the key distinguishing characteristics of the discipline of mathematics since the nineteenth century (Davis & Hersh, 1981). Indeed, Raman (2002), in her study, *Proof and Justification in Collegiate Calculus*, observes that "since the 6th century BC when Greek mathematicians established the axiomatic method, mathematicians have considered proof to be the *sine qua non* of mathematics" (p. 1). Proof is "the glue that holds mathematics together" (M. Atiyah as cited in Dunham, 1994, p. 15).

The term mathematical proof is not limited to a single definition: hence, it can be difficult to know, in any given context, exactly how the term is being used. The Oxford American Dictionary defines proof as "a demonstration of the truth of something" (1980, p. 535). Leddy (2001) offers one of the simplest and most practical definitions of proof: "a reasoned argument from acceptable truths" (p. 13). Yet, once one leaves simplistic definitions behind, the matter becomes more confusing. For example, a proof that is acceptable to a physicist might not be acceptable to a mathematician. Polya (1960) writes that

> in mathematics as in the physical sciences we may use observation and induction to discover general laws. But there is a difference. In the physical sciences, there is no higher authority than observation and induction, but in mathematics there is such an authority: rigorous proof.
>
> (as cited in Leddy, 2001, pp. 11-12)

In other words, as soon as there is sufficient evidence to support a scientist's hypothesis,—and as long as there is no evidence against it,—s/he accepts the hypothesis, but among most mathematicians, a claim to proof involves more stringent criteria. The mathematician reasons that observation cannot prove by itself because eyes can deceive us, measurement cannot prove because the certainty of the conclusion, depends upon the precision of the measuring instrument and the care of the measurer (both variable factors), and experiment cannot prove because the conclusions can only be considered probable and not invariable (Johnson, 2007).

Even within the mathematics community itself, standards of proof vary due to the autonomous development of mathematical specialties and their subsequent isolation from each other (Almeida, 1996). A number of key words, long used within the mathematics education literature to refer to elements of proof,—such as “explanation”, “verification”, and “justification”,—convey different meanings depending upon who is using them. This multiplicity of meaning implies fundamental differences in how mathematicians conceptualize proof. Specifically, definitions tend to vary according to the mathematician’s perception of what constitutes an “appropriate formal system” (Hanna, 1991, p. 55).

Obviously, mathematical proof is a complex matter both in terms of the multiplicity of definitions that have been offered to specify the concept and in the variety of functions that have been attributed to it. We see this complexity played out within educational contexts in a number of ways. How educators define proof and expect it to function depends upon the specific factors associated with the educational context including the teacher’s understanding of and experience with proof and the student’s age, grade level, and mathematical abilities.

Since the closing years of the nineteenth century, mathematicians have narrowly defined proof in terms of logic (Davis & Hersh, 1981; Gardiner & Moreira, 1999). Frege (1884/1950), for example, defined proof as a finite sequence of statements such that each statement in the sequence is either an axiom or a valid inference from previous statements. Many decades later, Alonzo Church (1956, as cited in Gardiner & Moreira, 1999) demonstrated the same adherence to formal logic. According to Church,

> a finite sequence of one or more well-formed formulas is called a proof if each of the well-formed formulas in the sequence either is an axiom or is immediately inferred from proceeding well-formed formulas in the sequence by means of one of the rules of inference. A proof is called a proof of the last well-formed formula in the sequence.
>
> (as cited in Gardiner & Moreira, 1999, p. 20)

Joseph (2000) provides a more recent and succinct take on proof as logical formalism. Proof, he claims, “is a procedure, [an] axiomatic deduction, which follows a chain of reasoning from the initial assumptions to the final conclusion” (p. 127).

Over the years, however, many mathematicians have come to define proof in broader terms. Almost thirty five years ago, Lakatos (1976) described mathematics as an open subject that is constantly being developed and changed through proofs and refutations. He suggested that the definition of proof should be expanded to include explanations, justifications and elaborations of any conjecture subjected to counter examples. Lakatos’ view reflects the assumption that proof depends on the insights of the active mathematician and not on mechanistic rules and procedures. Indeed, perceptions of what proof *is* have changed to such a degree that, little more than a decade ago, mathematician Thurston (1995) claimed, “for the present, formal proofs [in the sense of symbolic logic] are out of reach and mostly irrelevant” (p. 34). Even more inclusive is Hanna’s (1995) definition. She insists that the best proof is one that helps us understand the meaning of the theorem that is being proved. She notes that such proofs help us to see, not only *that* a theorem is true, but also *why* it is true. These, Hanna claims, are more convincing and more likely to lead to further discoveries. Hence, in school mathematics, the proofs that explain—narrative proofs—are much more important because they facilitate understanding (Hanna, 1990). In the end, whether one defines proof narrowly or broadly, it is important to remember that proof is an art and the act of proving can "evoke a profound sense of beauty and surprise" (Moreira, 1999, p. 349).

Given that mathematicians differ in their perception of what it is that constitutes a mathematical proof, it follows that they would also differ in their understanding of the role played by proof within mathematics. Indeed, one's view of what it is that proofs *do* typically influences how one defines the term. Various mathematicians and mathematics educators, Bell, 1976; de Villiers, 1990, 1999; Hanna, 2000; Hanna & Jahnke, 1996; Lucast, 2003; Luthuli, 1996; Marrades & Gutierrez, 2000; Leddy (2001), have identified the functions of proof and proving as: verification or justification (concerned with the truth of a statement); explanation (providing insight into why a statement is true); systemization (the organization of various results into a deductive system of axioms, major concepts and theorems); discovery (the discovery or invention of new results); communication (the transmission of mathematical knowledge); construction of an empirical theory; exploration of the meaning of a definition or the consequences of an assumption; incorporation of a well-known fact into a new framework, viewing it from a fresh perspective; providing an intellectual challenge to the author of the proof, and so on.

A Brief History of Proof

If one defines proof broadly, one can find evidence of mathematical proof in the extant computations of various cultural groups that pre-date the ancient Greeks. Of course, few would disagree with Szabo's (1972, as cited in Siu, 1993) assertion that the concept of deductive science was unknown to the eastern people of antiquity before the development of Greek culture. He maintains that

> in the mathematical documents which [have come] down to us from these [Eastern] people, there are no theorems or demonstrations and the fundamental concepts of deduction, definition and axiom have not yet been formed. These fundamental concepts made their first appearance in Greek Mathematics.
>
> (as cited in Siu, 1993, p. 345)

Indeed, if one defines "mathematical proof as a deductive demonstration of a statement based on clearly formulated definitions and postulates" (Siu, 1993, p. 345), then one must conclude that no proofs can be found in the surviving mathematical texts of the ancient Chinese, Indian, Egyptian or Babylonian peoples (Joseph, 2000). However, one does see within these texts a technical facility with computation, recognition of the applicability of certain procedures to a set of similar problems, and an understanding of the importance of verifying the correctness of a procedure (Joseph, 2000). If one defines proof generally as an explanatory note that serves to convince or enlighten the reader, then one can, in fact, identify an abundance of mathematical proofs and proving within these ancient texts. As Wilder (1978) reminds us, "we must not forget that what constitutes proof varies from culture to culture, as well as age to age" (p. 69).

The Greeks, in an attempt to lay solid foundations for geometry, were the first to introduce a version of the axiomatic method in mathematics (Hanna, 1983). Up to the time of Thales of Miletus (640-546 B.C.) two different ways to communicate mathematical statements were commonly used: illustrative examples that served as templates for a general statement or diagrams that made the statement obvious. The former told a reader how to obtain a result, while the latter helped the viewer internalize the idea by gaining an insight into why the idea was correct. Thales conceived of the need to reduce by logical argument the mathematical statements to simpler facts so as to make proof more convincing. He set out to prove geometric properties of figures by deduction rather than by measurement. By the time of the Pythagorean School (5^{th} century BC), proofs were quite well established, but mostly in paragraph form. Specifically, Euclid understood proof as a convincing argument based on intuitive truths in the human culture (Hanna, 1983).

Given the influence of the Greeks and geometry, the deductive approach in mathematics came to be referred to in the nineteenth century as the geometrical or Euclidean method. According to Grabiner (1974; cited in Hanna, 1983), during those years, a desire to focus and narrow mathematical results and avoid errors, as well as a need to formalize mathematical results, all played a part in stimulating a growing interest in formal proof. Throughout the century, perceptions of the role played by intuition in mathematics altered radically as mathematics went through several crises. Consequently, the notion of proof was further formalized. A modern variant, the two-column proof first appeared in Geometry textbooks about 1900. Two column proofs serve as a way to organize a series of statements (the left hand column) each one logically deduced from the previous one or based on definitions, axioms or previously proven theorem. Wu (1996) notes that “although the choice of Euclidean geometry as a starting point of proofs may have been an historical accident, it is nevertheless a felicitous accident” (p. 228) because most people learning to prove a proposition for the first time find it easier to look at a picture than to think abstractly. Over time, then, the Greek-inspired method of deductive proof came to play a central role in mathematics, though considering the lengthy history of mathematical thought and practices, a greater emphasis on rigor is a relatively recent phenomenon (Hanna, 1983).

Types of Proof

An awareness of the ways in which mathematicians have categorized proof can help in understanding the concept. Many classification systems have been put forth; however, there are four types of proof that mathematicians commonly identify.

1) *Proof by counter-example.* This type of proof involves finding at least one example in which a generalization is false. The counter-example will disprove the generalization or indicate its negation. A student, for instance, may conclude that a negative number plus a positive number is always a negative number, prompting another student to prove the conjecture wrong by offering a counter-example, say $-3 + 7 = 4$.

2) *Direct/Deductive proof.* In this case, one shows that a given statement is deducible by inferring patterns from given information, previously studied definitions, postulates and theorems. Traditionally, direct proofs have been expressed using two-column or paragraph formats. They can also be presented in the "flow-proof format" suggested by McMurray (1978).

3) *Indirect Proof.* With this type of proof, one assumes that the negation of a statement yet to be proven is true, then shows that this assumption leads to a contradiction. The following situation illustrates the process of an indirect argument. On arriving at the darkened library, Angela thinks, "The library must be closed". The logic behind her thought is this: When libraries are open, patrons and employees require light; thus, the lights are likely to be turned on. Right now, the lights are not

on; therefore, the library must be closed. Additionally, the process of proving a proof by proving its contra-positive can be thought of as a special case of indirect proof through contradiction. Paragraph formats are often used to show indirect proofs.

4) *Proof by induction.* According to O'Daffer and Thornquist (1993), this is the most complex type of proof. It is based on the principle of mathematical induction and can be stated as follows: If a given property is true for 1 and if for all $n > 1$, the property being true for n implies it is true for $n + 1$. Thus, we can conclude that the property is true for all natural numbers.

In addition, mathematicians have suggested various other ways of classifying students' justification and thought processes *while* they are involved in proving a mathematical task. Some of the well known classifications include those by van Hiele levels (Senk, 1989) and Sowder and Harel (1998). van Hiele's levels are exclusive to geometry. Although Sowder and Harels' classification is quite exhaustive, not all of their classifications can be applied to written works. The most comprehensive classification of proofs was put forward by Balaheff (1988, 1991). Balacheff identified four categories of proofs: (1) naïve empiricism, (2) crucial experiment, (3) generic example and (4) thought experiment. Balacheff situated his taxonomy within a developmental model of proving skills. He argued that each of these four levels of mathematical proof could be classified within one of two broad categories that he termed *pragmatic* justifications and *conceptual* justifications. He called all justifications *pragmatic* when they focused on the use of examples, actions or showings. He called justifications *conceptual* when they demonstrated abstract formulations of properties and relationships among properties.

Proof and Curriculum

Gardiner and Moreira (1999) claim that "mathematics is not proof; mathematics is not spotting patterns; mathematics is not calculation. *All are necessary, but none is sufficient*" (emphasis added; p. 19). Thus, they underscore that one cannot teach mathematics without teaching proof. Furthermore, Wu (1996) reminds us that the production of a statement is the basic methodology whereby we can ascertain whether the statement is true or not. He also notes that, any one who wants to know what mathematics is about must therefore learn how to write down a proof or at least understand what a proof is. Wu further elaborates by saying that it is in fact in the mathematics courses where the students get their rigorous training in logical reasoning. Also, in mathematics, students learn how to cut through deceptive trappings to get at the kernel of a provable fact, where they learn how to distinguish between what is provable and what is not. Hence, learning how to write proofs is a very important component in the acquisition of such skills.

In mathematics education, as Maher and Martino (1996) have argued, we are interested ultimately in student understanding, not just of mathematical principles but of the world itself, and proof and proving offer a means by which teachers might enhance student understanding. In fact, Marrades and Gutierrez (2000) insist that helping students "to [come to] a proper understanding of mathematical proof and [so] enhance their proof techniques" has become "one of the most interesting and difficult research fields in mathematics education" (p. 87).

Although proofs "are the guts of mathematics" (Wu, 1996, p. 222), unfortunately, many secondary school students have little experience and even less understanding of proof (Bell, 1976; Chazan, 1993; Hadas, Hershkowitz & Schwarz, 2000; Senk, 1985). To summarize these findings, one could say, proofs and proving have played a peripheral role at best in North American secondary school mathematics education (Knuth, 1999, 2002a, 2002b). Knuth (2002a) observes that teachers tend to introduce students to mathematical proof solely through the vehicle of Euclidean geometry in the US. Given this narrow application, it is not surprising that students develop little skill in identifying the objectives or functions of mathematical proof, or that both teachers and students come to perceive mathematical proof as a formal and meaningless exercise (Alibert, 1988; Knuth, 1999, 2002a, 2002b). In general, students learn to imitate and memorize specific proof structures by observing the teacher and studying the textbook, but fail to understand the diverse nature, function, and application of mathematical proof (Hadas, Hershkowitz & Schwarz, 2000). There is no doubt that proving is a complex task that involves a range of student competencies such as identifying assumptions, isolating given properties and structures and organizing logical arguments. If teachers wish to teach students to think for themselves, and not simply fill their minds with facts, then as Hanna and Jahnke (1996) stress, it is essential that they place greater emphasis on the communication of meaning rather than on the formal derivation. In this respect, the teaching and learning of mathematical proof appears to have failed. (Hadas, Hershkowitz & Schwarz, 2000).

Since 1989, the NCTM has called for substantive change in the nature and role of proof in secondary school mathematics curricula. The NCTM published the *Curriculum and Evaluation Standards for School Mathematics* (1989) at a time when the teaching of mathematical proof,—specifically within the US,— had almost disappeared from the curriculum or sunk into meaningless ritual (Knuth, 1999). In that document, the NCTM recommended that less emphasis be given to two-column proofs and to Euclidean geometry as an axiomatic system. In general, recommendations call for a shift in emphasis from (what has often been perceived as) an over-reliance on rigorous proofs to a conception of proof as convincing argument (Hanna, 1990). Unfortunately, the NCTM document encouraged educators and students to think that verification techniques could substitute for proof (Latterell, 2005). In that sense this document failed to utilize the broader perspectives of proof in the teaching and learning of mathematics. In contrast, a more recent NCTM document, *Principles and Standards for School Mathematics* (2000), identifies proof as an actual *standard* and assigns it a much more prominent role within the school mathematics curriculum. Accordingly, curriculum developers and program designers have come to expect that all students experience proof as an integral part of their mathematics education. Notably, the 2000 document recommends that reasoning and proof become a part of the mathematics curriculum *at all levels* from pre-kindergarten through grade 12. The section entitled *Reasoning and Proof* outlines for the reader that students should be able to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof. Given the greater status assigned to proof within the mathematics curriculum, it is essential that teachers plan curricular experiences that can help students develop an appreciation for the value of proof and for those strategies that will assist them in developing proving skills.

Any improvement in mathematics education for students depends upon effective mathematics teaching in the classroom. This includes providing students with opportunities to interact in the classroom, to propose mathematical ideas and conjectures, to evaluate personal thinking, and to develop reasoning skills. Teachers' knowledge and beliefs will play an important role in shaping students' understanding of math and their ability to solve mathematical problems (NCTM, 2000).

A Practical Example: Proof Categories

In order to help students develop proving skills, teachers must be able to anticipate the myriad ways in which students might answer a specific proving task. In the following section, I have identified a well-known proof problem and provide exemplars demonstrating how students might approach this problem.

Prove that the sum of the first *n* positive integers (S(n)) is $\frac{n(n+1)}{2}$.

Proof Level 1: Naïve Empiricism

Here the justification is based on the basis of a small number of examples. Balacheff (1988) calls this category of justification *naïve empiricism*.

1+2+3 = 6.
Here n=3.
S(n) = $\frac{n(n+1)}{2}$ = $\frac{3(3+1)}{2}$ = 6

1+2+3+4+5+6+7 = 28
Here n = 4.
S(n) = $\frac{n(n+1)}{2}$ = $\frac{7(7+1)}{2}$ = 28

Since these two work, sum of the first *n* positive integers (S(n)) is $\frac{n(n+1)}{2}$.

This is the lowest level in his proof taxonomy. Balacheff (1988) (as well as other researchers; for example, Knuth (1999)) does not consider this a valid proof. However, Balacheff, includes it in his hierarchy of proofs because students typically think that this is a valid proof.

Proof Level 2: Crucial Experiment

In this case, a conjecture is developed based on small number of examples and then that conjecture is verified using a larger number that is *intentionally* chosen.

1+2+3 = 6.

Here n=3.

S(n) = $\frac{n(n+1)}{2} = \frac{3(3+1)}{2} = 6$

1+2+3+4+5+6+7 = 28

Here n = 7.

S(n) = $\frac{n(n+1)}{2} = \frac{7(7+1)}{2} = 28$

Since these two work, I will try one more example with a larger number using my calculator. 1+2+3+ …+54 = 1485.

Here n= 54. S(n) = $\frac{54(54+1)}{2} = 1485$.

Since the assertion works in this case too, the sum of the first *n* positive integers (S(n)) is $\frac{n(n+1)}{2}$.

In fact, it is the *intentionalilty* that distinguishes naïve empiricism from crucial experiment. However, it is not always easy to distinguish between *naïve empiricism* and *crucial experiment* based on written work alone because, in both cases, the prover believes the conjecture proved on the basis of a small number of examples. Unless explicitly mentioned, it is easier to misread *crucial experiment* as *naïve empiricism*.

Proof Level 3: Generic Example

In generic example, a single example is specifically analyzed and this analysis attempts to generalize from this single case.

1+2+3+4+5+6+7 = 28

Here n = 7.

S(n) = 28, 28 = 7*4, 4 (the second number) comes from 8 $\div 2$ and 8 is obtained from 7+1 which n+1. So integers S(n) = $\frac{n(n+1)}{2}$. This is always the case.

This example illustrates generic reasoning because the calculations and answers are specific to the fact that one is considering 1+2+3+4+5+6+7, but the justification provided would apply whatever the *n* might be. In other words, *n* can assume any value. The example is a generalization of a class, not a specific example. Although the focus is once is on one case, it is as an example of a class of objects. The example is selected to perform operations and transformations to arrive at a justification. The operations and transformations are applied to the whole class.

Proof Level 4: Thought Experiment

At the level of thought experiment, the students are able to distance themselves from the action and make logical deductions based only upon an awareness of the properties and the relationships characteristic of the situation. Here, actions are internalized and dissociated from the specific examples considered. The justification is based on the use of transformation of formalized symbolic expressions.

I will provide two thought experiments. Both of these are adopted from Hanna (1990).

Thought Experiment – 1

For n=1 it is true since 1 = $\frac{1(1+1)}{2}$.

Assume it is true for some arbitrary *k*, that is S(*k*) = $\frac{k(k+1)}{2}$. Then consider

S(*k+1*) = S(*k*) + *k*+1

$$= \frac{k(k+1)}{2} + k+1.$$

$$= \frac{(k+1)(k+2)}{2}.$$

Therefore, the statement is true for *k*+1 if it is true for *k.* By induction, the statement is true for all *n.*

Thought Experiment – 2

This proof is similar to the well known Gauss formula.

Let S(*n*) = 1 + 2 + 3 + …..(*n*-1) + *n*

Then S(*n*) = *n* + (*n*-1) +(*n*-2) + ….. 2 + 1

Taking the sum of these two rows,

$$2\,S(n) = (1 + n) + [2 + (n-1)] + [3 + (n-2)] + [4 + (n-3)] +\ldots+ (n+1)$$

$$= (n+1) + (n+1) + (n+1) + \ldots+ (n+1)$$

$$= n(n+1)$$

Therefore, S(*n*) = $\frac{(n)(n+1)}{2}$.

It can be noted that in both these examples the arguments are *de-contextualized* from the specifics of an example to the generic aspects of the problem; that is, mental operations and logical deductions aim to validate the conjecture in a general way. Hanna distinguishes between thought experiments 1 and 2 as follows: 1 is a *proof that proves* and, 2, is a *proof that explains*. Note that the first proof, even though it proves the conjecture, does *not* provide insight into why the sum is $\frac{(n)(n+1)}{2}$. However, the second proof, not only proves it, but also reasons why the sum must be equal to $\frac{(n)(n+1)}{2}$.

The second proof *explains* (or reasons) why the conjecture is true using the symmetry (two different representations of the sum S(*n*)property.) This simple example indicates that there are proofs that convince us of the result, but do not explain it. Hanna underlines that this *explanatory value* of proofs is the most important aspect of proofs and proving in school mathematics given that the teacher's role is to "make student *understand* mathematics" (emphasis added; p. 12).

Explanatory Vs. Rigorous Proofs

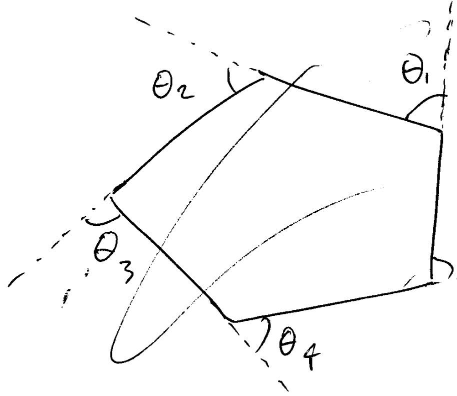
It is quite understandable that in school mathematics, the *explanatory* nature of proof would be deemed extremely important. I discuss below a response to a proof task, from Varghese (2007) that was submitted by a secondary school mathematics student teacher (student work follows the discussion). This student teacher's response demonstrates the assumption that an informal proof is automatically an explanatory proof. It is evident from this student teacher's diagram that he understands clearly what an exterior angle is. His scribbles indicate that he wanted to prove the task mathematically, rather than by verbal argument. It also seems that he tried to prove this task for a regular polygon since he produced drawings for the equilateral triangle, rectangle, regular pentagon, regular hexagon, and so on. He also provided a general formula for the size of an interior angle within a regular polygon. However, he got "stuck" and could not proceed. At that point, he likely resorted to the verbal argument.

According to Balacheff (1988, 1991)'s hierarchy, this narrative proof may be categorized as a verbal thought experiment.

Now, the question remains: is this an acceptable mathematical proof from the perspective of standards deemed appropriated for the high school level? Even more from a prospective mathematics teacher? I shared this proof with four secondary school mathematics teachers (in an informal discussion) and asked them whether or not it could be considered an acceptable mathematical proof. Three of the four teachers indicated that they would accept this as a valid mathematical proof and award it 100%. Indeed, they stressed that they found this an *interesting* way to prove the task. I am not generalizing from this observation, but it does raise the question: do we accept every convincing argument as a valid mathematical proof?

PLEASE COMPLETE THE FOLLOWING TASKS, IF TIME PERMITS.

d) Prove that the sum of the exterior angles of a polygon is always 360⁰.



~~This can be proved if we have the previous knowledge that, for any n-gon, $n \geq 3$ the size of the interior angle is~~

$\theta_{int} = \frac{(n-2)180}{n}$

~~The exterior angle is how m~~

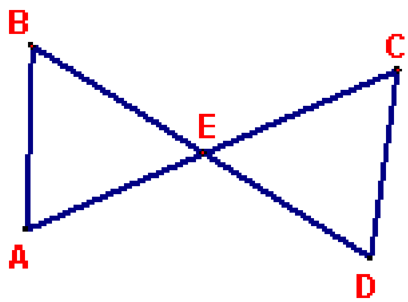
If you "walk" around the edge of a polygon, the exterior angle is a measure of how much you must turn to start walking on the next side. After doing a complete "walk" around the polygon you have done one complete circle which is 360°. ∴ the sum of the exterior angles must be 360°

In secondary school mathematics classrooms, teachers bear the responsibility of helping students effectively communicate using the language of mathematics. If we begin to accept any convincing argument as a mathematical proof assuming that all verbal arguments must be explanatory proofs, students are likely to confuse an informal convincing argument with a mathematical proof. This will lead students to the faulty conclusion that any argument can automatically constitute a complete mathematical proof. As Moreira and Gradiner (1999) notes, there is a danger in such view since the methods used to “convince” are selected in order to achieve an effect. However, when proving a statement in mathematics, one is restricted in the methods one can use and the reasons one can give as this involves definitions, axioms, and the propositions which have been previously proved.

Educators often complain that introducing “two-column” proofs at the secondary level is what puts students off; this is what compels them to protest, “I hate proof”. Watered- down treatment of mathematical proof may be why students find, in post secondary schools, the transition to ‘formal proof” extremely difficult (see Moore, 1994). Indeed, there are some mathematics educators who argue that secondary school is not the place for students to write rigorous, formal mathematical proofs, that teachers should postpone this work until post secondary school (see Wu, 1996). It seems that post secondary instructors assume that students in their classes will be able to handle formal proof (Moore, 1994; Wu, 1997). Given what is clearly the limited experience with formal proof that students have at the high school level, this would seem to be a false assumption.

Indeed, not only do university and college instructors assume that students will be able to deal with formal proofs, but they also typically overwhelm students with rigorous proofs (Moore, 1994; Knapp, 2005). Furthermore, university instructors’ expectations as they pertain to formalism in proof are commonly quite high (Moore, 1994). Wu (1997) notes that university instruction is based on the “Intellectual Trickle-down Theory of

Learning: aim the teaching at the best students, and somehow the rest will take care of themselves" (p.5). He also notes that university instruction and courses are "forward looking" (if you don't understand something in this course, you will understand it in the next course or courses). This "*far-better-things to come*" philosophy of university teaching (Wu, 1997, p. 7) and the fact that in secondary schools students lack sufficient experience with proof may very well explain why many mathematics students at the postsecondary level have unsophisticated understandings of mathematical proof. Varghese (2007) noted that student teachers have a limited facility in tackling mathematical proof tasks. Most of the student teachers' in that study shared that they had only a limited exposure to mathematical proof in secondary school, and yet at university they felt that they were overwhelmed with mathematical proof. Wu (1996) claims that since "we are in an age when mathematical knowledge is at a premium, it does not seem proper that correct mathematical reasoning should be suddenly declared too profound and too difficult for all high school students" (p. 224). I believe that high school students are capable of handling *formal* proofs as long as they do not experience an abrupt shift at Grade 10 or 11 from working with simplistic proofs to complex two-column proofs. One can easily transition from a *mathematically sound explanatory* proof to a formal two–column proof as long as the transition is carefully managed. Indeed, the same is true of the movement from two-column proof to mathematically sound explanatory proof. And, as I noted above, Balchaeff's taxonomy may be used as the vehicle for transition to mathematically sound proof. Teachers could generate examples as per Balacheff's hierarchy and teach students the fallacies associated with each of the lower level proofs. This may help to ensure that the transition to more formal proofs is a smooth one. From the following example one can easily note that it is quite easy to transition from a mathematically sound explanatory proof to a two-column proof.



If E is the midpoint of $\overline{BD}$ and $\overline{AE} \cong \overline{EC}$, Prove that $\Delta AEB \cong \Delta CED$

Explanatory Proof

We know that E is the midpoint of $\overline{BD}$ and that $\overline{AE} \cong \overline{EC}$. Since E is the midpoint of $\overline{BD}$, we know that $\overline{BE} \cong \overline{ED}$ because the midpoint of a segment divides the segment into two congruent segments. Since vertical angles are congruent, $\angle BEA \cong \angle DEC$. When two triangles have corresponding angles and sides that are congruent as above, the triangles are congruent according to SAS (Side, Angle, Side) congruence. Thus $\Delta AEB \cong \Delta CED$, according to SAS method.

	STATEMENT	REASON
1.	$\overline{AE} \cong \overline{EC}$	Given
2.	$\angle BEA \cong \angle DEC$	Vertical angles are congruent
3.	$\overline{BE} \cong \overline{ED}$	E is a midpoint of $\overline{BD}$ Midpoint divides the segment into two congruent segments
4	$\Delta AEB \cong \Delta CED$	If two sides and the included angle of one triangle are congruent to the corresponding

		parts of another triangle, the triangles are congruent according to SAS congruence.

Rather than introducing formal mathematical proofs as if they were suddenly plucked from thin air, teachers would do well to introduce students gradually to different ways of writing proofs, culminating, after careful and strategic planning, with two column proofs. Most of the difficulties that teachers anticipate when it comes to teaching proof may be resolved if they introduced mathematical reasoning and proving in elementary school and continued the process through more and more challenging examples all the way through secondary school. Such a practice would certainly be in line with the recommendations of the NCTM (2000). It is so unfortunate that many secondary students and, indeed, many secondary teachers regard two column proofs as 'the root of all evil'. In fact, two column proofs, as Wu (1996) asserts, make "most clear to a beginner what a mathematical proof really is: a connected sequence of assertions each backed up by a reason" (p. 227).

Summary

Without proof, Mathematics loses a great deal of its beauty. As Wu (1996) writes, Mathematics without proof is like "opera without [the] human voice". If we want to see prominence of proof in school mathematics curriculum, then teachers must be prepared to teach concepts of mathematical proof at all levels within the school system. If a developmental model such as Balacheff's taxonomy of proofs is applied in discussed in class, students will understand the fallacies of pragmatic proofs and they will be far more likely to move to conceptual proofs.

Explanatory function of proofs is clearly of great importance in an educational context, but it is also important that students learn how to communicate *correctly* using *correct* mathematical language: a verbal argument does not necessary constitute a mathematical proof. If watered-down proofs are accepted as the norm in school mathematics classrooms, students will have great difficulty understanding the scope and depth of the concept. We must remember that placing emphasis on the substantive part of mathematics (i.e. proofs) will help our students when they are expected to write proofs in real analysis, abstract algebra and advanced mathematical courses. For those students who go on to university, *rigorous* proofs will no longer seem so daunting; and for those university students who decide to become secondary school mathematics teachers, teaching mathematical proof in the classroom will no longer seem such an onerous and intimidating responsibility. As educators, we must do our best to prevent the “recycling effect” (Galbraith, 1982) that occurs when students who lack adequate understanding of critical concepts such as mathematical proof go on to become the next generation of teachers who are anxious, afraid, and reluctant to teach the critical concept of mathematical proof.

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