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This paper presents new findings on varieties generated by small semigroups and groups, and offers a survey of existing results. A companion website is provided which hosts a computational system integrating automated reasoning tools, finite model builders, SAT solvers, and GAP. This platform is a living guide to the literature. In addition, the first complete and justified list of identity bases for all varieties generated by a semigroup of order up to 4 is provided as supplementary

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Group
Identity
Identity basis

material. The paper concludes with an extensive list of open problems.

Contents

1. Introduction

1.1. Motivation

We assume familiarity with the general theory of varieties, semigroups, and groups. As references, we suggest the monographs of Almeida [1], Burris and Sankappanavar [10], Howie [38], McKenzie et al. [69], and H. Neumann [73].

Table 1
The semigroups U_1 and U_2.

U_1	1	2	3	4
1	1	1	3	3
2	2	2	4	4
3	1	1	3	3
4	2	2	4	4

U_2	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	2	2
3	1	1	1	3	3
4	1	1	3	4	5
5	1	3	3	4	5

The lattice of varieties of semigroups has been the subject of intensive investigation that started as early as the 1960s, but given its complexity, it remains an active area of research. There are several very good surveys, such as Evans [20], Gusev et al. [31], and Shevrin et al. [90], that allow the reader to become familiar with the main results and problems. Our goal is different: we aim to provide a living survey powered by a companion computational tool that helps the working mathematician find new results or locate known ones in the literature.

As an illustration, suppose that we need to investigate semigroups satisfying the implication

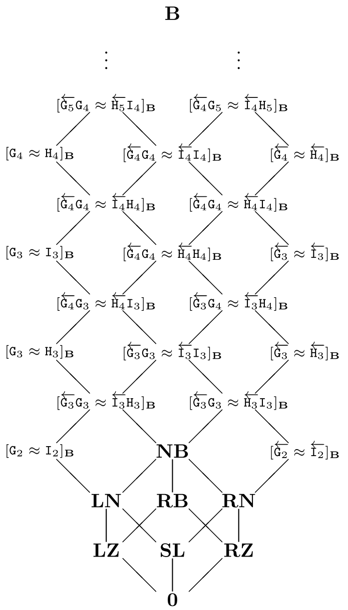
$$xy \approx yx \implies x \approx y,$$

objects we call *anti-commutative semigroups*. To understand their properties, we could use GAP [24] to find some small models, as for example, the semigroup U_1 in Table 1. At a certain point, we observe that all elements of U_1 are idempotents—such a semigroup satisfies the idempotency identity $x^2 \approx x$ and is commonly called a *band*—and searching for varieties of bands we find a reference [21] that contains the lattice $\mathscr{L}(\mathbf{B})$ of varieties of bands; see Fig. 1.

Again we could use GAP to see that our semigroup U_1 satisfies the identity $xyx \approx x$ but violates the identities $xy \approx x$ and $xy \approx y$. Therefore, the variety $\operatorname{var}\{U_1\}$ generated by U_1 is contained in the variety of bands defined by the identity $xyx \approx x$—the variety $\mathbf{RB}$ of *rectangular bands*—but is excluded from its two maximal subvarieties $\mathbf{LZ}$ and $\mathbf{RZ}$, whence $\operatorname{var}\{U_1\} = \mathbf{RB}$. Now an easy exercise shows that a semigroup is anti-commutative if and only if it satisfies the identity $xyx \approx x$, and from here we get access to an enormous amount of literature on our original object U_1. The key steps in the above process were the observation that U_1 is a band and the complete knowledge of the lattice of varieties of bands.

Now suppose that we are working with a different theory and our test semigroup is U_2 in Table 1. Since U_2 is not a band, there is no general lattice, similar to $\mathscr{L}(\mathbf{B})$ in Fig. 1, that allows us to repeat what we did with U_1. It turns out that the variety $\operatorname{var}\{U_2\}$ is defined by the identities

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx, \quad x_1^2x_2y_1^2y_2z^2 \approx y_1^2y_2x_1^2x_2z^2,$$



$$\mathbf{B} = [x^2 \approx x]$$
$$\mathbf{NB} = [xyzx \approx xzyx]_{\mathbf{B}}$$
$$\mathbf{LN} = [xyz \approx xzy]_{\mathbf{B}}$$
$$\mathbf{RN} = [xyz \approx zyx]_{\mathbf{B}}$$
$$\mathbf{SL} = [xy \approx yx]_{\mathbf{B}}$$
$$\mathbf{RB} = [xyx \approx x]$$
$$\mathbf{LZ} = [xy \approx x]$$
$$\mathbf{RZ} = [xy \approx y]$$
$$\mathbf{0} = [x \approx y]$$

Fig. 1. The lattice $\mathscr{L}(\mathbf{B})$ of varieties of bands, where $[\mathbf{u} \approx \mathbf{v}]_{\mathbf{B}} = \mathbf{B} \cap [\mathbf{u} \approx \mathbf{v}]$ and details on the words $\mathtt{G}_n, \mathtt{H}_n, \mathtt{I}_n, \overleftarrow{\mathtt{G}_n}, \overleftarrow{\mathtt{H}_n}, \overleftarrow{\mathtt{I}_n}$ are given in Section 2.4.

and this result is only available from Trahtman [98, Proposition 2], a paper published in Russian.

In general, given a semigroup S of order at most 6, there is a good chance that information on the variety var$\{S\}$ and its subvarieties can be found in the literature, since such varieties have received much attention over the years (see, for example, Lee [52], Lee et al. [62], Mel'nik [70], Perkins [78], Tishchenko [97], and Zhang and Luo [103]), especially in the investigation of the finite basis problem for small semigroups (see Lee et al. [60] and the references therein). The first goal of this survey is to provide such information, but we go far beyond that. The overall aim is twofold: survey identity bases defining varieties generated by finite semigroups; and provide a companion website [87], running in the background GAP, a SAT solver, automated reasoning tools, and finite model builders. The website will be periodically updated.

The following briefly describes the resources provided by the present survey and the website.

(A) Identity bases:

- (i) The website provides identity bases for all varieties generated by a semigroup of order up to 5. Proofs and references for varieties generated by a semigroup of order up to 4 are given in the supplementary material (Appendix A)—the first complete source providing such information. Information on varieties generated by a semigroup of order 5, given its relatively large volume, will be disseminated elsewhere.

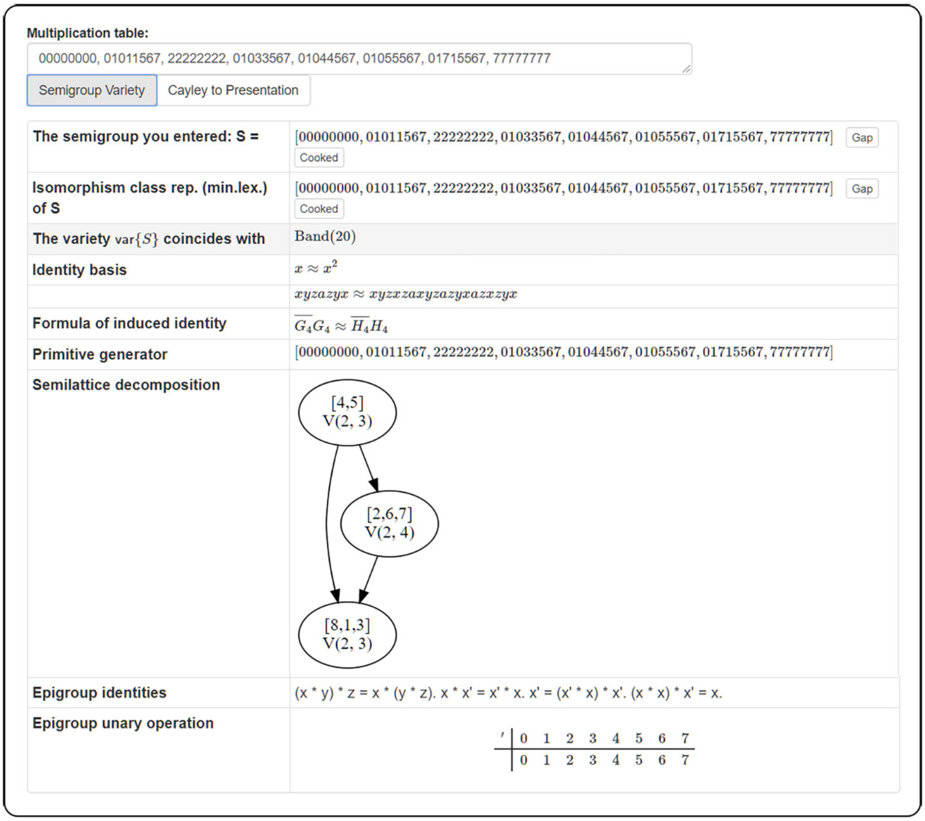


Fig. 2. Companion website: the variety generated by a band of order 8.

(ii) For some classes of semigroups, which include all bands, the website finds identity bases for varieties generated by an arbitrarily (but reasonably) large finite model; see Fig. 2.

(iii) The website provides identity bases for varieties generated by various groups, including all groups of order less than 24 and all metabelian groups G such that $\gcd(|N|, |G/N|) = 1$ for some abelian normal subgroup N of G. It also provides the maximal subgroups of the given semigroup and hence identity bases for them when available.

(iv) The website provides identity bases for the varieties covered by a given variety $\mathbf{V}$, provided that it is known. Therefore, the website can determine whether an input semigroup S generates a variety $\mathbf{V}$ by checking if S satisfies the identity basis for $\mathbf{V}$ and violates the identities defining the varieties covered by $\mathbf{V}$; see Section 4.1. The same procedure is used when varieties of groups are involved.

(v) Some semigroups S are non-finitely based and some are *inherently non-finitely based* in the sense that every locally finite variety containing S is non-finitely based. We list many results related to these two properties and provide the

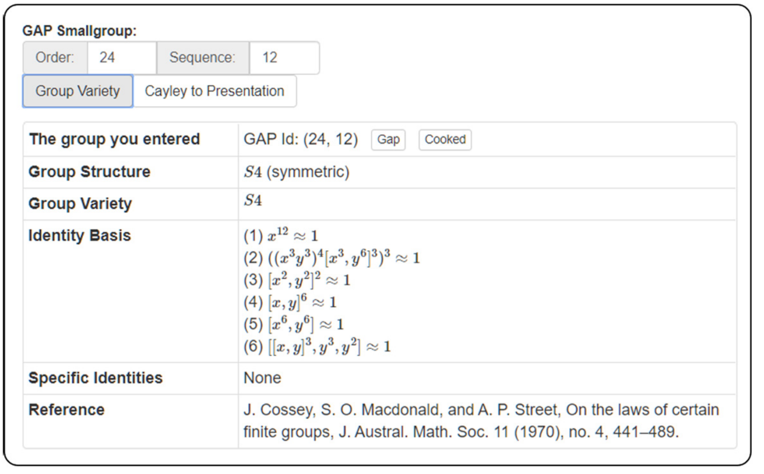


Fig. 3. Companion website: the variety generated by the symmetric group Sym_4.

first list of all inherently non-finitely based semigroups of order up to 8 (Theorem 4.8).

(vi) We also established the new results that proper subvarieties of var$\{\mathsf{Sym}_4\}$ and of var$\{\mathsf{SL}(2,3)\}$ are metabelian; see Fig. 3.

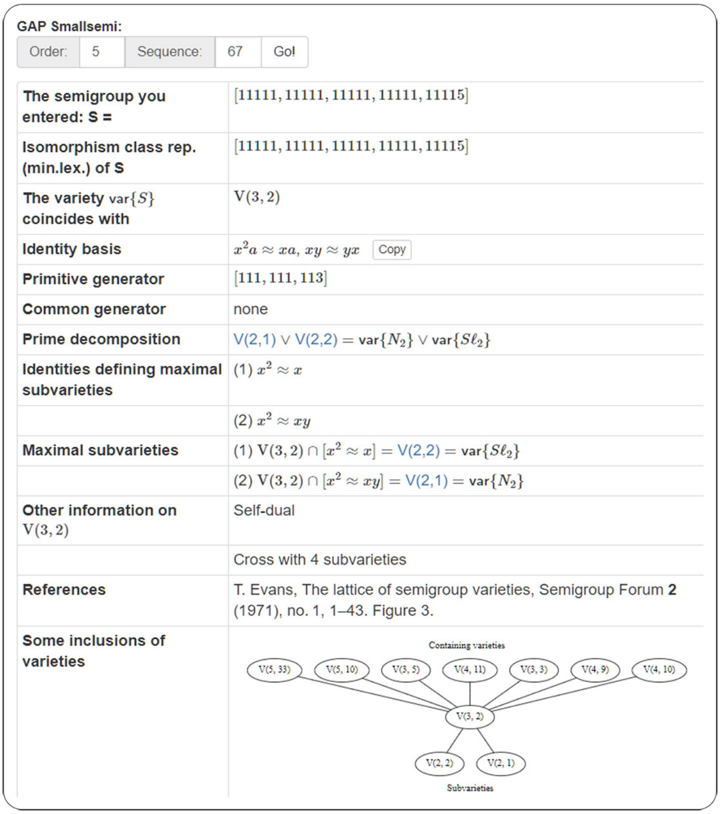
(B) To enter a semigroup or group into the website, the user has the option of inputing the multiplication table (in a very flexible way); GAP identifier; or a **V**-presentation, where **V** is any variety, quasivariety, or more generally, any class of algebras definable by first order formulas. The website also provides several tools to manipulate presentations.

(C) For many finite semigroups S, the companion website provides bibliographic information about the variety var$\{S\}$, such as its prime decomposition, the varieties that cover it, a generator of minimum order, and the semilattice decomposition of S and the varieties generated by the components; see Fig. 4.

1.2. Organization

Section 2 contains some background information on varieties of semigroups, groups, and epigroups; the lattice of varieties of bands; varieties with infinitely many subvarieties; and semilattice decompositions of semigroups.

Section 3 surveys some results on varieties generated by small groups; it consists of mostly known material and the main results here highlight the gaps in the literature waiting to be filled. It is our conviction that within the class of groups, the topic that is of most interest to us—explicit identity bases for varieties generated by small groups—was more or less *abandoned*, not because everything was too easy but exactly the opposite. Given the classification of finite simple groups, perhaps it is time for group theorists to start looking into varieties of groups again. It is worth mentioning that many closely



GAP Smallsemi:

Order:	5	Sequence:	67	Go!

The semigroup you entered: S =	$[11111, 11111, 11111, 11111, 11115]$
Isomorphism class rep. (min.lex.) of S	$[11111, 11111, 11111, 11111, 11115]$
The variety var$\{S\}$ **coincides with**	$\mathrm{V}(3,2)$
Identity basis	$x^2a \approx xa,\ xy \approx yx$ Copy
Primitive generator	$[111, 111, 113]$
Common generator	none
Prime decomposition	V(2,1) $\vee$ V(2,2) $=$ var$\{N_2\} \vee$ var$\{S\ell_2\}$
Identities defining maximal subvarieties	(1) $x^2 \approx x$
	(2) $x^2 \approx xy$
Maximal subvarieties	(1) $\mathrm{V}(3,2) \cap [x^2 \approx x] =$ V(2,2) $=$ var$\{S\ell_2\}$
	(2) $\mathrm{V}(3,2) \cap [x^2 \approx xy] =$ V(2,1) $=$ var$\{N_2\}$
Other information on $\mathrm{V}(3,2)$	Self-dual
	Cross with 4 subvarieties
References	T. Evans, The lattice of semigroup varieties, Semigroup Forum **2** (1971), no. 1, 1–43. Figure 3.
Some inclusions of varieties	

Fig. 4. Companion website: example of information displayed if the identity basis for the variety generated by the given semigroup is found.

connected areas are still very active, for instance, construction of short identities in symmetric groups (see Bulatov et al. [8], Karpova and Shur [46], and the references therein) and computational complexity of checking whether a given identity holds in a fixed finite group (see Burris and Lawrence [9], Földvári and Horváth [22], Horváth [34, 35], Horváth et al. [36], Horváth and Szabó [37], Kompatscher [48], and the references therein). For semigroup theorists, given a semigroup S, it might be useful to know to which varieties of groups belong the maximal subgroups (the $\mathcal{H}$-classes of idempotents) of S. In both this survey and the companion website [87], we make intensive use of the library of small groups [2,14,19,27,74,76] accessed through GAP and the library of the small semigroups [15] accessed through the GAP package Smallsemi [17].

Section 4 is concerned with an equational method to check if a variety is generated by a semigroup; non-finitely based semigroups of order 6; and inherently non-finitely based finite semigroups, in particular, a complete description of those that are of order up to 8.

The survey ends with a list of open problems in Section 5.

Information on all varieties generated by a semigroup of order up to 4 is given in the supplementary material (Appendix A).

Table 2
The semigroups J and J'.

J	1	2	3
1	1	1	1
2	1	1	1
3	1	2	3

J'	1	2	3
1	3	3	3
2	1	2	3
3	3	3	3

2. Preliminaries

2.1. Isomorphic semigroups and lexicographic minimum

Two algebras A and B of the same type are *isomorphic*, indicated by $A \cong B$, if there exists an isomorphism between them. The relation $\cong$ is an equivalence relation on any class of algebras of the same type. Occasionally, given a finite algebra A, it is practical to have a canonical representative of the equivalence class $[A]_{\cong}$. The following is a standard convention to determine such a representative when A is a semigroup.

Given a semigroup S of order n whose elements are labeled $1, 2, \ldots, n$, the *vector* of S, denoted by $\vec{v}(S)$, is the vector of dimension n^2 that is formed by concatenating the n rows of the multiplication table of S. For instance, the vector $\vec{v}(J)$ of the semigroup J in Table 2 is $[1, 1, 1, 1, 1, 1, 1, 2, 3]$. (For semigroups of order at most 9, it is unambiguous, and in fact clearer, to use commas only to separate different rows, for example, $\vec{v}(J) = [111, 111, 123]$.) The isomorphic copies of a given semigroup can then be lexicographically ordered as vectors. For instance, the semigroup J' in Table 2 is isomorphic to J, but since

$$\vec{v}(J) = [111, 111, 123] <_{\text{lex}} [333, 123, 333] = \vec{v}(J'),$$

we place J before J'.

For a semigroup S, an obvious choice for the representative of the class $[S]_{\cong}$ is the semigroup whose vector lexicographically precedes the vectors of all other semigroups in $[S]_{\cong}$. For instance, consider the semigroup

$$P = \langle a, b \mid ab = a,\ ba = 0,\ b^2 = b \rangle = \{0, a, b\}.$$

There are six semigroups on the set $\{1, 2, 3\}$ that are isomorphic to P, as shown in Table 3. Since $\vec{v}(S_1) \leq_{\text{lex}} \vec{v}(S_i)$ for all $i \neq 1$, the semigroup S_1 is the representative of the class $[P]_{\cong}$.

The *dual* of a semigroup S, denoted by $\overleftarrow{S}$, is the semigroup obtained from S by reversing its operation, that is, for every $a, b \in \overleftarrow{S} = S$, the product ab in $\overleftarrow{S}$ is equal to the product ba in S. The multiplication table of $\overleftarrow{S}$ is obtained simply by transposing the multiplication table of S. For instance, the semigroup $\overleftarrow{S_1}$ is isomorphic to the semigroup J in Table 2. The *dual* of a variety $\mathbf{V}$ is the variety

$$\overleftarrow{\mathbf{V}} = \{ \overleftarrow{S} \mid S \in \mathbf{V} \}.$$

Table 3
Semigroups isomorphic to P.

S_1	1	2	3
1	1	1	1
2	1	1	2
3	1	1	3

S_2	1	2	3
1	1	1	1
2	1	2	1
3	1	3	1

S_3	1	2	3
1	1	2	2
2	2	2	2
3	3	2	2

S_4	1	2	3
1	1	3	3
2	2	3	3
3	3	3	3

S_5	1	2	3
1	2	2	1
2	2	2	2
3	2	2	3

S_6	1	2	3
1	3	1	3
2	3	2	3
3	3	3	3

A variety $\mathbf{V}$ is *self-dual* if $\mathbf{V} = \overleftarrow{\mathbf{V}}$.

Two semigroups S and T are *equivalent* if either $S \cong T$ or $\overleftarrow{S} \cong T$. In the GAP package Smallsemi, semigroups are stored up to equivalence but not up to isomorphism, a decision with some disadvantages. In the present survey, unless otherwise stated, we work with semigroups up to isomorphism.

2.2. Varieties of semigroups

A class of algebras of the same type is a *variety* if it is closed under the formation of homomorphic images, subalgebras, and arbitrary direct products. The variety *generated by* a class $\mathbf{K}$ of algebras of the same type, denoted by $\operatorname{var}\mathbf{K}$, is the smallest variety containing $\mathbf{K}$; such a smallest variety exists because the intersection of varieties is a variety. A variety is *finitely generated* if it is generated by a single finite algebra. For any finitely generated variety $\mathbf{V}$ of semigroups, there exist only finitely many non-isomorphic generators of minimum order, say $S_1, S_2, \ldots, S_k$ with

$$\vec{\mathrm{v}}(S_1) <_{\mathrm{lex}} \vec{\mathrm{v}}(S_2) <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} \vec{\mathrm{v}}(S_k).$$

Then S_1 is the *primitive generator* of $\mathbf{V}$.

By a theorem of Birkhoff [4], a class $\mathbf{V}$ of algebras of the same type is a variety if and only if it coincides with the class of algebras satisfying some set Σ of identities; in this case, the variety $\mathbf{V}$ is *defined by* Σ and Σ is an *identity basis* for $\mathbf{V}$. A variety is *finitely based* if it possesses a finite identity basis. Since an algebra satisfies the same identities as the variety it generates, it is unambiguous to define an *identity basis* for an algebra A to be an identity basis for $\operatorname{var}\{A\}$ and say that A is *finitely based* whenever $\operatorname{var}\{A\}$ is finitely based. Every semigroup of order at most 5 is finitely based, but there exist semigroups of order 6 that are non-finitely based; see Section 4.2.

It is clear that if two algebras A and B are isomorphic, then $\operatorname{var}\{A\} = \operatorname{var}\{B\}$; the converse, however, does not hold in general. For instance, the dihedral group D_4 and the quaternion group Q_8 are non-isomorphic groups of order 8 that generate the same variety [73, 54.23].

Table 4
Some data on semigroups of order $n \leq 9$.

n	Number of semigroups of order n up to equivalence	Number of semigroups of order n up to isomorphism	Number of varieties with a primitive generator of order n
1	1	1	1
2	4	5	5
3	18	24	14
4	126	188	53
5	1,160	1,915	145
6	15,973	28,634	At least 463
7	836,021	1,627,672	Unknown
8	1,843,120,128	3,684,030,417	Unknown
9	52,989,400,714,478	105,978,177,936,292	Unknown

Up to isomorphism, the number of semigroups of order at most 5 is 2,133, while the number of varieties generated by such a semigroup is only 218; see Table 4 [16, Tables 1 and 3].

2.3. Varieties of groups

For a general reference on varieties of groups, we recommend the monograph of H. Neumann [73].

Two sets of identities that define the same variety are *equivalent*. Unlike the case for semigroups, every variety generated by a finite group has a finite identity basis, and every finite set of group identities is equivalent to a single identity. Therefore, every variety generated by a finite group can be defined by a single identity. We will see a similar phenomenon for varieties of bands in Section 2.4.

More details on varieties of groups can be found in Section 3.

2.4. The lattice of varieties of bands

A description of the lattice $\mathscr{L}(\mathbf{B})$ of varieties of bands can be found in Birjukov [3], Fennemore [21], Gerhard [25], Gerhard and Petrich [26], and Howie [38]; see Fig. 1. At the very top of the lattice is the variety $\mathbf{B} = [x^2 \approx x]$ of all bands. In the lower region is the sublattice $\mathscr{L}(\mathbf{NB})$ of $\mathscr{L}(\mathbf{B})$ consisting of eight varieties:

$$\mathbf{NB} = [xyzx \approx xzyx]_{\mathbf{B}}, \qquad \text{normal bands;}$$
$$\mathbf{LN} = [xyz \approx xzy]_{\mathbf{B}}, \qquad \text{left normal bands;}$$
$$\mathbf{RN} = [xyz \approx yxz]_{\mathbf{B}}, \qquad \text{right normal bands;}$$
$$\mathbf{SL} = [xy \approx yx]_{\mathbf{B}}, \qquad \text{semilattices;}$$
$$\mathbf{RB} = [xyx \approx x], \qquad \text{rectangular bands;}$$
$$\mathbf{LZ} = [xy \approx x], \qquad \text{left zero bands;}$$
$$\mathbf{RZ} = [xy \approx y], \qquad \text{right zero bands;}$$

$$\mathbf{0} = [x \approx y], \qquad \text{trivial semigroups.}$$

The remaining varieties in the lattice $\mathscr{L}(\mathbf{B})$ are defined by identities that are formed by the words $\mathtt{G}_n, \mathtt{H}_n, \mathtt{I}_n$, $n \geq 2$, inductively defined as follows:

$$\begin{aligned} &\mathtt{G}_2 = x_2x_1, && \mathtt{H}_2 = x_2, && \mathtt{I}_2 = x_2x_1x_2, \\ \text{and} \quad &\mathtt{G}_n = x_n\overleftarrow{\mathtt{G}_{n-1}}, && \mathtt{H}_n = \mathtt{G}_nx_n\overleftarrow{\mathtt{H}_{n-1}}, && \mathtt{I}_n = \mathtt{G}_nx_n\overleftarrow{\mathtt{I}_{n-1}}, \quad \text{for all } n \geq 3, \end{aligned}$$

where $\overleftarrow{W}$ is the word W written in reverse. For example,

$$[\mathtt{G}_3 \approx \mathtt{H}_3]_{\mathbf{B}} = [x_3x_1x_2 \approx x_3x_1x_2x_3x_2,\, x^2 \approx x].$$

By simple inspection of the identities in Fig. 1, it is clear that the varieties in column 3 are self-dual, the varieties in columns 1 and 5 are dual to each other, and the varieties in columns 2 and 4 are dual to each other.

The variety generated by a band B is the variety $\mathbf{V}$ of bands that satisfies both of the following properties: B belongs to $\mathbf{V}$ and B is excluded from every maximal subvariety of $\mathbf{V}$. When a semigroup S is entered into the companion website, there is a first test to check if S is a band. In the affirmative case, the website crawls up the lattice in Fig. 1; the first identity satisfied by S defines the variety $\operatorname{var}\{S\}$.

2.5. Varieties with infinitely many subvarieties

A variety that contains only finitely many subvarieties is *small*. It easily follows from the well-known theorem of Oates and Powell [75] that every finite group generates a small variety of semigroups. But this result does not hold in general. A counterexample is the monoid N_2^1 obtained by adjoining an identity element to the nilpotent semigroup

$$N_2 = \langle a \mid a^2 = 0 \rangle = \{0, a\};$$

see Fig. 5. Not only is the variety $\operatorname{var}\{N_2^1\}$ not small [20], it is the only non-small variety among all varieties generated by a semigroup of order 3 or less; see the supplementary material (Appendix A).

Properties more extreme than being non-small can be satisfied by a variety generated by a semigroup of order greater than 3. For instance, there exist

- semigroups of order 4 that generate a variety that is *finitely universal* in the sense that its lattice of subvarieties embeds every finite lattice [51];
- semigroups of order 6 that generate finitely universal varieties with continuum many subvarieties [18,40].

Every variety with continuum many subvarieties discovered so far is also finitely universal. It is unknown if there exists a variety of semigroups with continuum many

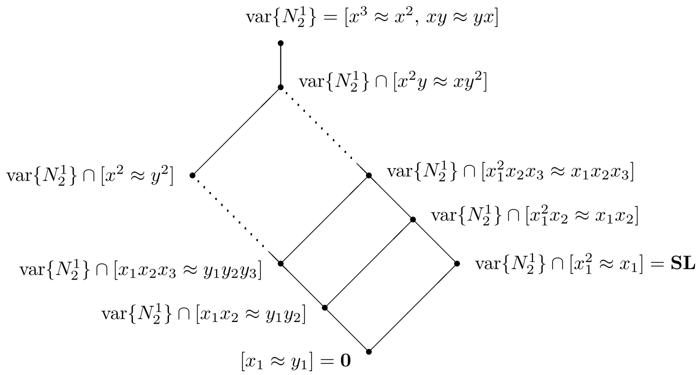


Fig. 5. The lattice of subvarieties of var$\{N_2^1\}$.

subvarieties that is not finitely universal. Refer to Shevrin et al. [90] for a survey of results regarding other properties satisfied by lattices of varieties of semigroups.

Given a finite semigroup, it is of natural interest to determine if it generates a small variety. Whether or not smallness of a variety is decidable remains open, but one special case is known. An identity of the form

$$x_1x_2\cdots x_n \approx x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)},$$

where π is some nontrivial permutation on $\{1,2,\ldots,n\}$, is a *permutation identity.* A nontrivial identity of the form $x_1x_2\cdots x_n \approx \mathbf{w}$ that is not a permutation identity is *diverse.*

Proposition 2.1 *(Malyshev [67]). Every variety of semigroups that satisfies some permutation identity and some diverse identity is small.*

2.6. Varieties of epigroups

An element a of a semigroup S is an *epigroup element of index n* if n is the least positive integer such that a^n belongs to a subgroup of S, that is, the $\mathcal{H}$-class H_{a^n} of a^n is a group; epigroup elements of index one are *completely regular.* If e is the identity element of H_{a^n}, then ae is in H_{a^n} and we define the *pseudo-inverse* of a by $a' = (ae)^{-1}$, where $(ae)^{-1}$ denotes the inverse of ae in the group H_{a^n} [89, Section 2.1]. An *epigroup* is a semigroup consisting entirely of epigroup elements, and a *completely regular semigroup* is a semigroup whose elements are all completely regular. An important result is that all finite semigroups are examples of epigroups. Following Petrich and Reilly [79] for completely regular semigroups and Shevrin [89] for epigroups, it is now customary to consider an epigroup or a completely regular semigroup $(S,\cdot)$ as a unary semigroup $(S,\cdot,{}')$, where $x \mapsto x'$ is the map that sends each element to its pseudo-inverse.

For any semigroup S, let $\mathrm{Epi}(S)$ denote the set of all epigroup elements of S of any index and let $\mathrm{Epi}_n(S)$ denote the subset of $\mathrm{Epi}(S)$ consisting of elements of index bounded by n. Then the inclusions

$$\mathrm{Epi}_1(S) \subseteq \mathrm{Epi}_2(S) \subseteq \cdots \subseteq \bigcup\{\mathrm{Epi}_n(S) \mid n \geq 1\} = \mathrm{Epi}(S)$$

hold, where $\mathrm{Epi}_1(S)$ consists of completely regular elements of S, and $\mathrm{Epi}(S) = S$ if and only if S is an epigroup.

For any $a \in \mathrm{Epi}_n(S)$, let e_a denote the identity element of the group H_{a^n}. Then $ae_a = e_a a$ is in H_{a^n} and the definition of pseudo-inverse introduced above leads to a characterization of the epigroup elements of the semigroup: $a \in \mathrm{Epi}(S)$ if and only if there exist some $n \geq 1$ and some (necessarily unique) $a' \in S$ such that

$$a'aa' = a', \quad aa' = a'a, \quad a^{n+1}a' = a^n; \tag{2.1}$$

see Shevrin [89, Section 2]. If a is an epigroup element, then so is a' with $a'' = aa'a$. The element a'' is always completely regular and $a''' = a'$. A standard notation in finite semigroup theory is to write $a^\omega = aa'$ for an epigroup element a; see, for example, Almeida [1]. Then

$$a^\omega = a''a' = a'a'', \quad (a')^\omega = (a'')^\omega = a^\omega,$$

and more generally, for any $m \geq 1$,

$$a^\omega = (aa')^m = (a')^m a^m = a^m (a')^m.$$

For each $n \geq 1$, the class $\mathbf{E}_n$ consisting of all epigroups S such that $S = \mathrm{Epi}_n(S)$ is a variety; in particular, $\mathbf{E}_1$ is the class of completely regular semigroups. The variety $\mathbf{E}_n$ is defined by the identities

$$xx' \approx x'x, \quad x(x')^2 \approx x', \quad x^{n+1}x' \approx x^n.$$

It is clear that the inclusions $\mathbf{E}_1 \subset \mathbf{E}_2 \subset \mathbf{E}_3 \subset \cdots$ hold and are proper, and that any variety of epigroups is contained in $\mathbf{E}_n$ for all sufficiently large $n \geq 1$ [89].

2.7. Semilattice decomposition of semigroups

There are several methods where a semigroup can be decomposed into smaller subsemigroups, for example, direct products, subdirect products, and Zappa–Szép extensions. Some have the property that each component cannot be further decomposed using the same method, in which case the decomposition is *atomic*. An obvious example of atomic decompositions for finite algebras is the direct product decomposition as—resorting to an argument similar to the one used to prove that every natural number is a product of prime numbers—we can easily show that every finite algebra can be decomposed into a direct product of directly indecomposable algebras. Finding atomic decompositions of infinite semigroups is more difficult; according to Bogdanović et al. [5], there are only five known atomic decompositions of general semigroups: semilattice decomposition [94], ordinal decomposition [66], U-decomposition [88], orthogonal decomposition [6], and the general subdirect decomposition whose atomicity was proved by Birkhoff.

In this survey, we will concentrate on semilattice decompositions of semigroups. A *semilattice* is a partially ordered set $(Y, \leq)$ in which every pair $i, j \in Y$ of elements has a greatest lower bound $i \wedge j$ in Y, called the *meet* of i and j. A semigroup S is a *semilattice of semigroups* if there exist a semilattice $(Y, \leq)$ and a family $\{S_i \mid i \in Y\}$ of semigroups indexed by Y such that $S = \bigcup\{S_i \mid i \in Y\}$ and $S_iS_j \subseteq S_{i\wedge j}$. Every semigroup can be decomposed into a semilattice of semigroups $\{S_i \mid i \in Y\}$ with each S_i being semilattice indecomposable [94].

It is easy to prove that every semilattice Y induces a commutative semigroup of idempotents and conversely. Therefore, the term *semilattice* is commonly used to refer to a commutative band or a partially ordered set admitting meet of every pair of elements.

Tamura [95] provided two equivalent ways to find the smallest semilattice congruence. For any semigroup S, let S^1 denote the smallest monoid containing S, that is,

$$S^1 = \begin{cases} S & \text{if } S \text{ is not a monoid,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

Then the smallest semilattice decomposition of S is the smallest partition containing the sets $\big\{\{xy, yx, xyx\} \,\big|\, (x, y) \in S^1 \times S^1\big\}$.

3. Varieties of groups

The theory of varieties of groups differs from that of semigroups in several ways, which will be briefly mentioned here. In particular, after a decade of activities around the 1960s, the monograph of H. Neumann [73] was published; this is still the best reference for the subject, and its notation became standard among group theorists. But one deviation is necessary in the present survey: varieties of groups in H. Neumann [73] are denoted by Fraktur capital letters, such as $\mathfrak{A}$ for the variety of abelian groups; for the sake of consistency, we will use bold-face letters such as $\mathbf{A}$ instead.

3.1. The basics

Since the symbols A_0 and A_2 are reserved for important semigroups in this survey (see Section 4.2) and in the companion website [87], to avoid confusion, the alternating group over n symbols is denoted by Alt_n. Similarly, since S_n is often used to represent semigroups, the symmetric group over n symbols is denoted by Sym_n. Other important finite groups required in the present survey include the cyclic group $\mathbb{Z}_n$ of order n, the dihedral group D_n of order $2n$, the quaternion group Q_8 of order 8, the special linear group $\mathsf{SL}(n, q)$ of degree n over a field of order q, and the projective special linear group $\mathsf{PSL}(2, q)$ of degree 2 over a field of order q.

As briefly noted in Section 2.3, every finite group has an identity basis that consists of one single group identity. Every group identity is equivalent to one of the form $\mathbf{w} \approx 1$, where $\mathbf{w}$ is a word in the variables and their inverses. We can regard $\mathbf{w}$ as an element of the free group $F(X)$ over a countable set X of variables. The identities satisfied by a variety form a fully invariant subgroup of $F(X)$, one mapped into itself by all endomorphisms of the group. Hence there is a bijection between varieties of groups and fully invariant subgroups of $F(X)$.

Each finite group of finite exponent $e \geq 2$ satisfies the identity $x^e \approx 1$ and so also the identity $x^{e-1} \approx x^{-1}$. Therefore, any identity of a finite group G is equivalent to one of the form $\mathbf{w} \approx 1$, where $\mathbf{w}$ is a semigroup word. In fact, a more specific result holds. Recall that the *derived subgroup* of G is the subgroup generated by all *commutators* $[g, h] = g^{-1}h^{-1}gh$ with $g, h \in G$. A *commutator word* is an element of the derived subgroup of the free group. Alternatively, a commutator word can be described as one in which the sum of the exponents of every variable is 0.

Theorem 3.1 *(B. H. Neumann [72]). Every identity of a finite group of exponent e is equivalent to $\{x^e \approx 1, \mathbf{w} \approx 1\}$ for some commutator word $\mathbf{w}$.*

A *factor* of a group G is a quotient of a subgroup of G, that is, H/K where $K \trianglelefteq H \leq G$; it is *proper* unless $H = G$ and $K = \{1\}$. A *chief factor* is one where $K \trianglelefteq G$ and H/K is a minimal normal subgroup of G/K; a *composition factor* is a factor H/K, when H and K are subnormal in G (that is, terms in a descending series in which each term is normal in its predecessor) and K is a maximal normal subgroup of H.

For any subgroups $A, B \leq G$, let $[A, B]$ denote the subgroup of G generated by the commutators in $\{[a, b] \mid a \in A,\, b \in B\}$. The *lower central series* is the descending series

$$G = G_1 > G_2 > G_3 > \cdots$$

with $G_{i+1} = [G_i, G]$. If c is the smallest integer such that $G_{c+1} = \{1\}$, then G is *nilpotent of class c*. The *derived series* is the descending series

$$G = G^{(0)} > G^{(1)} > G^{(2)} > \cdots$$

with $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. If ℓ is the smallest integer such that $G^{(\ell)} = \{1\}$, then G is *solvable of derived length ℓ*.

The *product* $\mathbf{UV}$ of varieties $\mathbf{U}$ and $\mathbf{V}$ consists of all groups G that are *extensions* of a group $H \in \mathbf{U}$ by a group $K \in \mathbf{V}$, that is, G has a normal subgroup isomorphic to H

with quotient isomorphic to K. The product of two varieties is a variety, and the product operation is associative. But product varieties are not usually generated by finite groups.

Theorem 3.2 *(Šmel'kin [92]). A product of three or more nontrivial varieties is not generated by a finite group. A product* $\mathbf{UV}$ *is generated by some finite group if and only if* $\mathbf{U}$ *and* $\mathbf{V}$ *have coprime exponents,* $\mathbf{U}$ *is nilpotent, and* $\mathbf{V}$ *is abelian.*

The variety $\mathbf{UV}$ has an identity basis of the form $\mathbf{u}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) \approx 1$, where $\mathbf{u}(x_1, x_2, \ldots, x_n) \approx 1$ is an identity of $\mathbf{U}$ and each $\mathbf{v}_i \approx 1$ is an identity of $\mathbf{V}$. (Note that, even if we can do better for some cases, usually all identities of $\mathbf{V}$ are needed, not just an identity basis.) We will give more details in an important special case in Section 3.3.

Many other results about varieties of groups are known, but our interest lies in those that are finitely generated.

One important result on varieties of groups is the Oates–Powell theorem, asserting that every finite group generates a finitely based variety. The result is actually much stronger. A variety is *Cross* if it is finitely based, finitely generated, and small.

Theorem 3.3 *(Oates and Powell [75]). The variety generated by any finite group is Cross.*

A finite group is *critical* if it does not belong to the variety generated by all its proper factors. Every locally finite variety of groups is generated by its critical groups [73, 51.41]. However, this result is not true for varieties that are not locally finite [77].

Sometimes one critical group is enough to generate the variety but sometimes it is not. For example, the variety $\mathbf{A}_6$ of abelian groups of exponent dividing 6 has $\mathbb{Z}_2$ and $\mathbb{Z}_3$ as critical groups and both are needed to generate the variety. On the other hand, the product $\mathbf{A}_2\mathbf{A}_2$ is generated by any one of its critical groups D_4 and Q_8.

If a group G contains a unique minimal normal subgroup N, then G is *monolithic* and N is the *monolith* of G. Two non-isomorphic critical groups that generate the same variety have abelian monoliths; it follows that non-isomorphic finite simple groups cannot generate the same variety [73, 53.35].

3.2. Abelian groups

The structure of the lattice of varieties generated by abelian groups is very easy to describe. The class $\mathbf{A}$ of all abelian groups is the variety defined by the identity $[x, y] \approx 1$; for each integer $m \geq 1$, the class $\mathbf{A}_m$ of abelian groups of exponent dividing m is the variety defined by the identities $x^m \approx [x, y] \approx 1$. Hence the lattice of varieties of abelian groups is isomorphic to the set of positive integers ordered by divisibility with a top element adjoined. We remark that GAP includes commands `IsAbelian` and `Exponent`, so the inclusion $\operatorname{var}\{G\} \subseteq \mathbf{A}_m$ is easily checked.

Checking the reverse inclusion $\mathbf{A}_m \subseteq \operatorname{var}\{G\}$ is more problematic. For sufficiently large m, there are continuum many varieties of groups covering $\mathbf{A}_m$ [39,50].

3.3. Metabelian groups

A group is *metabelian* if it lies in the product variety $\mathbf{AA}$, that is, it has an abelian normal subgroup with abelian quotient. Among small groups, many are metabelian; for example, 1,005 of the 1,048 groups of order at most 100 are metabelian. The smallest non-metabelian groups are the groups Sym_4 and $\mathsf{SL}(2,3)$ of order 24.

Every finite metabelian group belongs to the variety $\mathbf{A}_m\mathbf{A}_n$ for some $m, n \geq 1$. The smallest subgroup of a group G whose quotient is abelian of exponent dividing n is generated by the nth powers and commutators in G, so the variety $\mathbf{A}_m\mathbf{A}_n$ is defined by the identities

$$x^{mn} \approx [x, y]^m \approx [x^n, y^n] \approx \big[x^n, [y, z]\big] \approx \big[[x, y], [z, w]\big] \approx 1.$$

However, finding an identity basis for individual finite metabelian groups is more difficult. To this end, the commutator of $n \geq 3$ elements—the *left-normed commutator*—is required:

$$[x_1, x_2, \ldots, x_{n-1}, x_n] = \big[[x_1, x_2, \ldots, x_{n-1}], x_n\big].$$

Higman [33] showed that for each prime p that does not divide $n \geq 1$, the proper subvarieties of $\mathbf{A}_p\mathbf{A}_n$ containing $\mathbf{A}_{pn}$ are characterized by an identity of the form

$$[x^n, y^{d_1}, y^{d_2}, \ldots, y^{d_k}] \approx 1,$$

where $d_1 > d_2 > \cdots > d_k \geq 1$ are divisors of n such that d_i does not divide d_j whenever $i > j$.

As an example which we will examine later, consider the subvariety $\mathrm{var}\{\mathsf{Alt}_4\}$ of $\mathbf{A}_2\mathbf{A}_3$. The only possible Higman identity is $[x^3, y] \approx 1$, which does not hold in Alt_4. Therefore, $\mathrm{var}\{\mathsf{Alt}_4\} = \mathbf{A}_2\mathbf{A}_3$.

H. Neumann [73, 54.42] quotes a generalization of this, an unpublished result of C. H. Houghton according to which, assuming that $\gcd(m, n) = 1$, any such variety lies between $\mathbf{A}_{rs}$ and $\mathbf{A}_r\mathbf{A}_s$ for some $r, s \geq 1$ such that r divides m and s divides n. Moreover, such a variety is defined by identities of the form

$$[x^s, y^{d_1}, \ldots, y^{d_k}]^t \approx 1,$$

where t is a divisor of r and $d_1 > d_2 > \cdots > d_k \geq 1$ are divisors of n such that d_i does not divide d_j whenever $i > j$. See Mikaelian [71] for a proof of this result and for a generalization that determines when the equality $\mathrm{var}\{A\}\,\mathrm{var}\{B\} = \mathrm{var}\{A \wr B\}$ holds for abelian groups A and B with $A \wr B$ being their wreath product.

There are also results for the case when the condition $\gcd(m, n) = 1$ is relaxed. An important example, which consists of dihedral groups, is discussed below.

For an example, consider `SmallGroup(12,1)` in GAP with presentation

$$\langle a, b \,|\, a^3 = 1,\, b^4 = 1,\, b^{-1}ab = a^2 \rangle.$$

Clearly, this group belongs to $\mathbf{A}_3\mathbf{A}_4$. Since $\gcd(3, 4) = 1$, this group can be handled with Higman's theorem. The possible Higman identities are $[x^4, y] \approx 1$ and $[x^4, y^2] \approx 1$. It is readily shown that the second is satisfied but the first is not. Adding $[x^4, y^2] \approx 1$ to the identity basis we see that the identity $[x^4, y^4] \approx 1$ is redundant and can be discarded. Further reductions are possible, but we do not strive for the simplest identity basis.

A result of Kovács [49] describes the variety generated by a finite dihedral group. We restate his theorem in a format that is more useful for us.

Theorem 3.4. *Let $n = 2^d m$ with $d \geq 0$ and m odd.*

(i) *If $d \leq 1$, then* $\operatorname{var}\{\mathsf{D}_n\} = \mathbf{A}_m\mathbf{A}_2$.
(ii) *If $d \geq 2$ and $m = 1$, then* $\operatorname{var}\{\mathsf{D}_n\} = \mathbf{A}_{2^{d-1}}\mathbf{A}_2 \cap \mathbf{N}_d$, *where $\mathbf{N}_d$ is the variety of nilpotent groups of class at most d.*
(iii) *If $d \geq 2$ and $m > 1$, then* $\operatorname{var}\{\mathsf{D}_n\} = \operatorname{var}\{\mathsf{D}_{2^d}, \mathsf{D}_m\}$.

Now it follows from our general remarks on metabelian groups that an identity basis for $\mathbf{A}_n\mathbf{A}_2$ is given by $x^{2n} \approx [x^2, y^2] \approx 1$ (since a group belongs to $\mathbf{A}_n\mathbf{A}_2$ if and only if the squares commute and have orders dividing n). An identity basis for $\mathbf{N}_d$ is given by $[x_1, x_2, \ldots, x_{d+1}] \approx 1$. Given varieties $\mathbf{V}$ and $\mathbf{W}$, an identity basis for $\mathbf{V} \cap \mathbf{W}$ consists of the union of the identity bases for $\mathbf{V}$ and $\mathbf{W}$. Finally, the identities of $\operatorname{var}\{G, H\}$ consist of all products of an identity of G and an identity of H (in disjoint sets of variables). So the identities of varieties of dihedral groups can be described explicitly.

3.4. Other groups

Apart from the groups considered so far, results on particular finite groups are scarce. Cossey and Macdonald [11] and Cossey et al. [12] found explicit identity bases for the varieties $\operatorname{var}\{\mathsf{Sym}_4\}$, $\operatorname{var}\{\mathsf{Alt}_5\}$, and $\operatorname{var}\{\mathsf{PSL}(2, 7)\}$; they also found identities that hold in $\operatorname{var}\{\mathsf{PSL}(2, p^m)\}$ with prime p and $m \geq 1$ but without proof that these identities form an identity basis. An identity basis for $\operatorname{var}\{\mathsf{PSL}(2, 2^m)\}$ with $m \geq 2$ was found by Southcott [93].

A description of the identities of the group $\mathsf{SL}(2, q)$ is also available in some cases: $q = 9$ or $q = p^m$ for some odd prime $p \not\equiv \pm 1 \pmod{16}$ and odd $m \geq 1$. In these cases, the identities are of the form $[\mathbf{w}, x] \approx 1$ and $\mathbf{w}^2 \approx 1$, where $\mathbf{w} \approx 1$ ranges over an identity basis for $\mathsf{PSL}(2, q)$ and x is a variable not occurring in $\mathbf{w}$. In particular, this result holds for $\mathsf{SL}(2, 3)$ and $\mathsf{PSL}(2, 3) \cong \mathsf{Alt}_4$, where identities of the latter group have been described in Section 3.3.

Cossey et al. [12, Theorem 3.6] claim to have an identity basis for the variety $\operatorname{var}\{\mathsf{Sym}_5\}$, but the identity

$$\left\{\left((x^{25}y^{25})^{36}\cdot\left\{(x^{35}y^{25})^{50}\cdot(x^{25}y^{35})^{50}\right\}^{36}\right)^{36}\cdot\left[\left[[x^{25},y^{25}]^{15},y^{15}\right]^{25},y^{50}\right]\right\}^{5}\approx 1$$

they gave, as observed by Leedham-Green and O'Brien [65], does not hold in Sym_5. (However, if the last exponent 5 is replaced with 15, then the resulting identity holds in Sym_5.) We have not found a correct identity basis for var$\{\mathsf{Sym}_5\}$.

Regarding the variety var$\{\mathsf{PSL}(2,5)\}$, an identity basis was first found by Cossey and Macdonald [11], but one of its identities involve 61 variables. Shortly after, an identity basis that involves only 2 variables was exhibited by Bryant and Powell [7].

3.5. Non-metabelian groups of order 24

As noted earlier, Sym_4 and $\mathsf{SL}(2,3)$ are the only non-metabelian groups of order 24. An identity basis for the variety var$\{\mathsf{Sym}_4\}$ is in Cossey et al. [12]:

$$x^{12}\approx\left((x^3y^3)^4[x^3,y^6]^3\right)^3\approx[x^2,y^2]^2\approx[x,y]^6\approx[x^6,y^6]\approx\left[[x,y]^3,y^3,y^2\right]\approx 1.$$

Our goal here is to describe the subvarieties of the varieties var$\{\mathsf{Sym}_4\}$ and var$\{\mathsf{SL}(2,3)\}$, and to show each of their proper subvarieties is metabelian.

Lemma 3.5. *Let G be any non-abelian group in* var$\{\mathsf{Sym}_3\}$*. Then G has a subgroup isomorphic to* Sym_3 *and so* var$\{G\}=$ var$\{\mathsf{Sym}_3\}$*.*

Proof. We know that G' is a nontrivial elementary abelian 3-group while G/G' is an abelian group that is a direct product of elementary abelian 2-groups and 3-groups. Since G is not abelian, there exist $a,b\in G$ that fail to commute. We consider various cases, assuming that there is no subgroup isomorphic to Sym_3 and aiming for a contradiction. Note that any two elements of order 3 commute, since each of them is a square and $[x^2,y^2]\approx 1$ is an identity of Sym_3.

- $|a|=|b|=2$. Then $\langle a,b\rangle$ is a dihedral group of order 6 or 12 and so contains a subgroup isomorphic to Sym_3. So we may assume that involutions commute.
- $|a|=2$ and $|b|=3$. Then $c=b^a$ is another element of order 3 and c commutes with b. Since $(bc^{-1})^a=cb^{-1}=(bc^{-1})^{-1}$, the subgroup $\langle a,b\rangle$ is isomorphic to Sym_3. Hence we can assume that elements of prime orders commute.
- $|a|\in\{2,3\}$ and $|b|=6$. Then a commutes with b^2 and b^3, and so with b.
- $|a|=|b|=6$. Then a^2 and a^3 both commute with b, so a and b commute.

The proof is thus complete. □

Theorem 3.6. *Let G be any critical group in* var$\{\mathsf{Sym}_4\}$ *that is not metabelian. Then* var$\{G\}=$ var$\{\mathsf{Sym}_4\}$*.*

Proof. Let N be the verbal subgroup of G corresponding to the identities of var$\{\mathsf{Sym}_3\}$, that is, the subgroup generated by values in G of the identities of Sym_3. Then N is an elementary abelian 2-group, and it is nontrivial because $\{1\} \neq G'' \leq N$. Further, G/N belongs to var$\{\mathsf{Sym}_3\}$.

If G/N is abelian, then $G' \leq N$, so the contradiction $G'' = \{1\}$ is deduced. Therefore, G/N is non-abelian. Further, G/N has order divisible by 3, since otherwise G is a 2-group; but 2-groups in var$\{\mathsf{Sym}_4\}$ belong to var$\{\mathsf{D}_4\}$ and so are metabelian. Therefore, by Lemma 3.5, the group G/N must contain a subgroup K isomorphic to Sym_3.

Moreover, such a subgroup in G/N cannot centralize N. For if it did, then the centralizer $C_G(N)$ of N (and hence G) would have a normal 3-subgroup; but G is critical and therefore monolithic (it contains a unique minimal normal subgroup, which is a 2-group) [73, 51.32].

An orbit of K on N has order at most 6 and so generates a subgroup of order at most 2^6. We show that there must be such a subgroup of order 2^2. First, consider the action of an element of order 3 in K; let $\{x_1, x_2, x_3\}$ be an orbit. The subgroup $\langle x_1, x_2, x_3\rangle$ has order 2^2 or 2^3; in the latter case, the subgroup $\langle x_1x_2, x_2x_3, x_3x_1\rangle$ has order 2^2.

If such a subgroup $\{1, y_1, y_2, y_3\}$ of order 2^2 is invariant under an element t of order 2 in K, then our claim is proved; so suppose not. Let $z_i = y_i^t$ where $i \in \{1, 2, 3\}$. Then the group $\langle y_i, z_i \mid i = 1, 2, 3\rangle$ has order 2^4 and is invariant under Sym_3. We can assume that conjugation by an element u of order 3 in K induces the permutation $(y_1, y_2, y_3)(z_1, z_3, z_2)$ (since t inverts u). Then the subgroup $\langle y_1z_1, y_2z_3, y_3z_2\rangle$ has order 2^2 and is Sym_3-invariant.

Now the group generated by K together with this K-invariant subgroup of N is isomorphic to Sym_4, and belongs to var$\{G\}$. So var$\{\mathsf{Sym}_4\} \subseteq$ var$\{G\}$, and we have equality as required. □

Corollary 3.7. *Any proper subvariety of* var$\{\mathsf{Sym}_4\}$ *is metabelian.*

The analogous result for $\mathsf{SL}(2, 3)$ is similar but easier to establish. We have noted in Section 3.3 that the identities of $\mathsf{SL}(2, 3)$ have the form $[\mathbf{w}, x] \approx \mathbf{w}^2 \approx 1$, where $\mathbf{w} \approx 1$ ranges over the identities of Alt_4 and x is a variable not in $\mathbf{w}$.

Theorem 3.8. *Let G be any critical group in* var$\{\mathsf{SL}(2, 3)\}$ *that is not metabelian. Then* var$\{G\}$ = var$\{\mathsf{SL}(2, 3)\}$.

Proof. The preliminary result that a non-abelian group in var$\{\mathsf{Alt}_4\}$ contains a subgroup isomorphic to Alt_4 is proved in a manner similar to the analogous result for Sym_3.

Let $G \in$ var$\{\mathsf{SL}(2, 3)\}$ be critical and non-metabelian. Then G'' is an elementary abelian 2-group and is contained in the center $Z(G)$, so all its subgroups are normal in G. Since G is monolithic, $|G''| = 2$. Now G/G'' has a subgroup isomorphic to Alt_4, and it is easy to see that this lifts to a subgroup of G isomorphic to $\mathsf{SL}(2, 3)$. □

Corollary 3.9. *Any proper subvariety of* var$\{\mathsf{SL}(2, 3)\}$ *is metabelian.*

3.6. Small groups

We now demonstrate that we have covered all groups of order less than 24. We have seen that identity bases for abelian groups are trivial. The remaining groups are metabelian, and in most cases Higman's theorem applies. We note, for example, that the two non-abelian groups of order 8—the dihedral group D_4 and quaternion group Q_8—generate the same variety [73, 54.23].

The outstanding cases are the three groups of order 16 whose derived subgroups are cyclic of order 4:

- the dihedral group
$$\mathsf{D}_8 = \langle a, b \mid a^8 = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle;$$
- the *semi-dihedral group*
$$\mathsf{SD}_8 = \langle x, y \mid x^8 = y^2 = 1,\ y^{-1}xy = x^3\rangle;$$
- the *generalized quaternion group*
$$\mathsf{Q}_{16} = \langle p, q \mid p^8 = 1,\ q^2 = p^4,\ q^{-1}pq = p^{-1}\rangle.$$

We sketch a proof that these three groups generate the same variety, whose description follows from Kovács [49]; see Theorem 3.4. (Details of the calculations in this proof were carefully checked with GAP.)

Take two copies of the dihedral group, the second generated by elements c and d satisfying the same relations as a and b; let G be the *central product* of these two groups (the quotient by the central subgroup generated by a^4c^4). Now D_8 is a subgroup of G, and G is a quotient of a direct product of two copies of D_8, so that $\mathrm{var}\{\mathsf{D}_8\} = \mathrm{var}\{G\}$.

Now we can find copies of the other two groups in G as follows:

- $\mathsf{Q}_{16} = \langle p, q\rangle$, where $p = a$ and $q = bc^2$;
- $\mathsf{SD}_8 = \langle x, y\rangle$, where $x = ac^2$ and $y = b$.

Clearly $q^2 = c^4 = p^4$, and the other relations of Q_{16} are clear; and $y^{-1}xy = a^{-1}c^2 = a^3c^6 = x^3$, and the other relations of SD_8 are clear. Therefore, $\mathrm{var}\{\mathsf{Q}_{16}\}$ and $\mathrm{var}\{\mathsf{SD}_8\}$ are subvarieties of $\mathrm{var}\{\mathsf{D}_8\}$.

To show equality in the first case, we note that G has an automorphism φ interchanging a and c and also b and d. The image of $\langle p, q\rangle$ under φ is $\langle c, a^2d\rangle$. Now these groups commute, since $bc^2 \cdot a^2d = c^2a^{-2}bd$ while $a^2d \cdot bc^2 = a^2c^{-2}bd$, and we have $c^2a^{-2} = a^2c^{-2}$ since $a^4 = b^4$. Moreover, it is easy to see that they are distinct. So they generate their central product, which is G. Thus G, and hence D_8, belongs to $\mathrm{var}\{\mathsf{Q}_{16}\}$.

The same method does not work for the second case since the two copies of SD_8 in G do not commute; so we reverse the argument. Take two copies of SD_8, generated by x, y and z, w, respectively, and let H be their central product (the quotient of the direct product by $\langle x^4z^4\rangle$). Put $a = xz^2$ and $b = y$. Then $a^8 = b^2 = 1$ and

$$b^{-1}ab = y^{-1}xz^2y = x^3z^2 = x^7z^{-2} = (xz^2)^{-1},$$

so that $\langle a, b\rangle$ is isomorphic to D_8. Therefore, D_8 is a subgroup of H, so that $\mathrm{var}\{\mathsf{D}_8\} \subseteq \mathrm{var}\{H\} = \mathrm{var}\{\mathsf{SD}_8\}$, whence we have equality.

3.7. Toward an explicit bound

It follows from the Oates–Powell theorem (Theorem 3.3) that the variety generated by a finite group is Cross. Can explicit bounds on the orders of critical groups in such a variety be extracted from the proof of this result?

The proof of the Oates–Powell theorem rests on three lemmas: the third concerns the class $\mathbf{C}(e, m, c)$ of finite groups of exponent e, whose chief factors are of order at most m and whose nilpotent factors have class at most c. Then $\mathbf{C}(e, m, c)$ is a class of finite groups in a variety such that for all $G \in \mathbf{C}(e, m, c)$, every critical group in $\mathrm{var}\{G\}$ belongs to $\mathbf{C}(e, m, c)$.

Lemma 3.10 *(H. Neumann [73, 52.23]). The class* $\mathbf{C}(e, m, c)$ *contains only finitely many critical groups up to isomorphism.*

Lemma 3.11 *(H. Neumann [73, page 156]). Suppose that* $G \in \mathbf{C}(e, m, c)$ *is any critical group with non-abelian monolith. Then* $|G| \leq m!$.

The abelian monolith case is much harder. Let $\Phi(G)$ denote the intersection of all maximal subgroups of G, called the *Frattini subgroup* of G. As H. Neumann [73, page 156] says:

> If a bound for the index of $\Phi(G)$ in G is found, then a bound for $|G|$ can be derived. For, since $\Phi(G)$ consists of all non-generators of G, the number of elements needed to generate G can be at most $|G/\Phi(G)|$. But from bounds for the number of generators of G and the index of $\Phi(G)$ in G, one obtains a bound for the number of generators of $\Phi(G)$ by means of Schreier's formula. As $\Phi(G)$ is nilpotent, of class at most c and exponent dividing e, this leads to a bound for the order of $\Phi(G)$, and so for the order of G.

Suppose we can show that $|G/\Phi(G)| \leq b$. Then G has at most $\log_2 b$ generators, so our bound for the number of generators of $\Phi(G)$ is $1 + (b-1)\log_2 b$, or in broad brush terms, $d \leq b \log b$. This gives a bound for the order of $\Phi(G)$ which is roughly $\exp(d + d^2 + \cdots + d^c)$, since the lower central factors are generated by commutators.

A small improvement is possible. If $\Phi(G)$ is not a p-group, then it is the direct product of its Sylow p-subgroups, each of which contains a nontrivial normal subgroup of G, contradicting the fact that G is monolithic. So we can replace e in the above bound by the largest prime divisor of e.

Continuing, the proof considers a series

$$\Phi(G) < F < C < G,$$

and shows that $|G/C| \leq (m!)^c$ and $|F/\Phi(G)| \leq m^c$, while $|C/F| \leq (m!)^t$, where $t \leq 1 + ce(m!)$. The bound for b is the product of these numbers.

Even for very moderate values of e, m, and c, the resulting bound is typically large.

4. Varieties of semigroups

4.1. Identity bases for finitely generated varieties

Let $\mathbf{V}$ be any finitely generated variety. Then the number of maximal subvarieties of $\mathbf{V}$ is some positive integer $k \geq 1$ [61, Proposition 4.1]; let $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_k$ be these maximal subvarieties. By maximality, each $\mathbf{M}_i$ can be defined within $\mathbf{V}$ by some identity μ_i. If $k \geq 2$, then $\mathbf{V} = \mathbf{M}_i \vee \mathbf{M}_j$ for all distinct i and j; if $k = 1$, then $\mathbf{M}_1$ is the unique maximal subvariety of $\mathbf{V}$ and so $\mathbf{V}$ is said to be *prime*. It follows that each finitely generated variety is either prime or a join of some of its prime subvarieties.

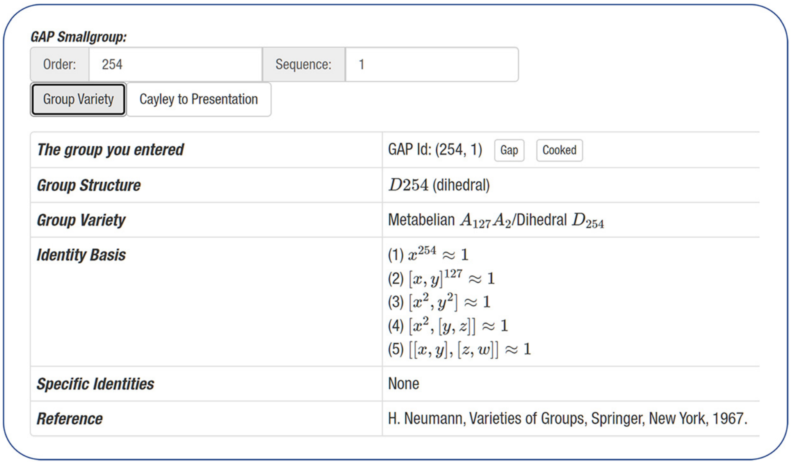
Regardless of the value of $k \geq 1$, it is easily seen that for any semigroup S, the equality $\mathbf{V} = \mathrm{var}\{S\}$ holds if and only if $S \in \mathbf{V}$ and $S \notin \mathbf{M}_i$ for all i. Further, if the variety $\mathbf{V}$ is finitely based and a finite identity basis Σ is available, then the equality $\mathbf{V} = \mathrm{var}\{S\}$ holds whenever $S \models \Sigma$ and $S \not\models \mu_i$ for all i. Therefore, the system $(\Sigma; \mu_1, \mu_2, \ldots, \mu_k)$ of identities, called a *Bas-Max system* for $\mathbf{V}$, provides an easily verifiable sufficient condition to check if a finite semigroup S generates $\mathbf{V}$.

Presently, the website database contains Bas-Max systems for many varieties, which include all of the following:

(a) varieties with a primitive generator of order at most 4;
(b) proper subvarieties of Cross varieties in (a);
(c) varieties with a primitive generator of order 5.

If a semigroup S entered into the website is shown to generate a variety $\mathbf{V}$ via its Bas-Max system $(\Sigma; \mu_1, \mu_2, \ldots, \mu_k)$, then the website reports the identity basis Σ for $\mathrm{var}\{S\}$, and other important information including the primitive generator for $\mathbf{V}$, any decomposition of $\mathbf{V}$ into a join of its prime subvarieties, and in many cases, the number of subvarieties of $\mathbf{V}$.

Bas-Max systems for varieties in (a) and (b), together with the aforementioned properties, will be established in the supplementary material (Appendix A). Justification of



The group you entered	GAP Id: (254, 1) Gap Cooked
Group Structure	$D254$ (dihedral)
Group Variety	Metabelian $A_{127}A_2$/Dihedral D_{254}
Identity Basis	(1) $x^{254} \approx 1$ (2) $[x, y]^{127} \approx 1$ (3) $[x^2, y^2] \approx 1$ (4) $[x^2, [y, z]] \approx 1$ (5) $[[x, y], [z, w]] \approx 1$
Specific Identities	None
Reference	H. Neumann, Varieties of Groups, Springer, New York, 1967.

Fig. 6. Companion website: information for GAP smallgroup (254,1).

the Bas-Max systems for varieties in (c), due to their relatively large volume, will be disseminated elsewhere.

The website will be regularly updated with newly established Bas-Max systems for varieties.

For groups of order at most 255, the website displays any known information; see, for example, Fig. 6.

4.2. Non-finitely based semigroups of order 6

Every variety generated by a semigroup of order at most 5 is finitely based [53,98]. Among all varieties generated by a semigroup of order 6, precisely four are non-finitely based [59,64]; these varieties are generated by the following semigroups:

- the monoid B_2^1 obtained from the Brandt semigroup

$$B_2 = \langle a, b \mid a^2 = b^2 = 0,\ aba = a,\ bab = b \rangle = \{0, a, b, ab, ba\};$$

- the monoid A_2^1 obtained from the 0-simple semigroup

$$A_2 = \langle a, b \mid a^2 = aba = a,\ bab = b,\ b^2 = 0 \rangle = \{0, a, b, ab, ba\};$$

- the semigroup A_2^g obtained by adjoining a new element g to A_2, where multiplication involving g is given by $g^2 = 0$ and $gx = xg = g$ for all $x \in A_2$;
- the $\mathscr{J}$-trivial semigroup

$$L_3 = \langle a, b \mid a^2 = a,\ b^2 = b,\ aba = 0 \rangle = \{0, a, b, ab, ba, bab\}.$$

Table 5
Non-finitely based semigroups of order 6.

B_2^1	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	2	2	3
3	1	2	3	1	3	1
4	1	1	1	4	4	6
5	1	2	3	4	5	6
6	1	4	6	1	6	1

A_2^1	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	2	2	3
3	1	2	3	2	3	3
4	1	1	1	4	4	6
5	1	2	3	4	5	6
6	1	4	6	4	6	6

A_2^g	1	2	3	4	5	6
1	1	1	1	1	1	6
2	1	1	1	2	3	6
3	1	2	3	2	3	6
4	1	1	1	4	5	6
5	1	4	5	4	5	6
6	6	6	6	6	6	1

L_3	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	1	1	1	3
4	1	1	2	1	4	2
5	1	1	3	1	5	3
6	1	2	2	4	4	6

The multiplication tables of these semigroups are given in Table 5; refer to Lee et al. [60] for more information on their discovery.

Remark 4.1. For semigroups endowed with additional unary operations, such as involution semigroups and restriction semigroups, there exist examples with fewer than six elements that generate non-finitely based varieties [23,45].

Besides the four non-finitely based semigroups of order 6, many other non-finitely based finite semigroups have been discovered since the 1970s; see Volkov [101]. But explicit identity bases have not been found for varieties generated by most of these semigroups because the task is neither necessary (in establishing the non-finite basis property) nor trivial. Nevertheless, explicit identity bases are available for a few examples.

Proposition 4.2 *(Jackson [42, Proposition 4.1]). The identities*

$$x^4 \approx x^3, \quad x^3y \approx yx^3, \quad x^2yx \approx x^3y, \quad xyx^2 \approx x^3y, \quad xyxzx \approx x^3yz,$$
$$\Big(\prod_{i=1}^{m} x_i\Big)\Big(\prod_{i=m}^{1} x_i\Big)y^2 \approx y^2\Big(\prod_{i=1}^{m} x_i\Big)\Big(\prod_{i=m}^{1} x_i\Big), \quad m = 1, 2, 3, \ldots$$

constitute an identity basis for a non-finitely based variety generated by a certain semigroup of order 211*.*

Proposition 4.3 *(Lee and Volkov [63, Section 1]). For each $n \geq 2$, the identities*

$$x^{n+2} \approx x^2, \quad (xy)^{n+1}x \approx xyx, \quad xyxzx \approx xzxyx,$$
$$\Big(\prod_{i=1}^{m} x_i^n\Big)^3 \approx \Big(\prod_{i=1}^{m} x_i^n\Big)^2, \quad m = 2, 3, 4, \ldots$$

constitute an identity basis for the non-finitely based variety $\mathrm{var}\{A_2, \mathbb{Z}_n\}$*. In particular,* $\mathrm{var}\{A_2, \mathbb{Z}_2\} = \mathrm{var}\{A_2^g\}$.

Proposition 4.4 *(Lee [54, Corollary 3.5]). For each $n \geq 1$, the identities*

$$x^{n+2} \approx x^2, \quad x^{n+1}yx^{n+1} \approx xyx, \quad xhykxty \approx yhxkytx,$$
$$x\left(\prod_{i=1}^{m}(y_i h_i y_i)\right)x \approx x\left(\prod_{i=m}^{1}(y_i h_i y_i)\right)x, \quad m = 2, 3, 4, \ldots$$

constitute an identity basis for the non-finitely based variety $\mathrm{var}\{L_3, \mathbb{Z}_n\}$*. In particular,* $\mathrm{var}\{L_3, \mathbb{Z}_1\} = \mathrm{var}\{L_3\}$.

4.3. Inherently non-finitely based finite semigroups

The *finite basis problem*—first posed by Tarski [96] in the 1960s as a decision problem—asks which finite algebras are finitely based. This problem is undecidable for general algebras [68] but remains open for semigroups. In contrast, it is decidable if a finite semigroup S is *inherently non-finitely based* in the sense that every locally finite variety containing S is non-finitely based. This result follows from the work of Sapir [80,81], a description of which requires the following important concepts:

- the *period* of a semigroup S is the least number d such that S satisfies the identity $x^{m+d} \approx x^m$ for some $m \geq 1$;
- the *upper hypercenter* of a group G, denoted by $\Gamma(G)$, is the last term in the upper central series of G;
- a word $\mathbf{w}$ is an *isoterm* for a semigroup S if for any word $\mathbf{w}'$ that is different from $\mathbf{w}$, one has $S \not\models \mathbf{w} \approx \mathbf{w}'$;
- the *Zimin words* $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$ are words over $\{x_1, x_2, x_3, \ldots\}$ defined inductively by $\mathbf{z}_1 = x_1$ and $\mathbf{z}_{k+1} = \mathbf{z}_k x_{k+1} \mathbf{z}_k$ for each $k \geq 1$.

Theorem 4.5 *(Sapir [83, Theorem 3.6.34]).*

(i) *A finite semigroup S is inherently non-finitely based if and only if there exists some idempotent $e \in S$ such that the submonoid eSe of S is inherently non-finitely based.*
(ii) *A finite monoid M with period d is inherently non-finitely based if and only if there exist $a \in M$ and an idempotent $e \in MaM$ such that the elements eae and $ea^{d+1}e$ do not belong to the same $\Gamma(M_e)$-coset of M_e, where M_e is the maximal subgroup of M containing e.*
(iii) *A finite semigroup S is inherently non-finitely based if and only if the Zimin words $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_m$, where $m = |S|^3$, are isoterms for S.*

The non-finitely based semigroups A_2^g and L_3 are not inherently non-finitely based because $\mathbf{z}_2 = x_1x_2x_1$ is not an isoterm; they satisfy the identities $\mathbf{z}_2 \approx x_1(x_2x_1)^3$ and $\mathbf{z}_2 \approx x_1x_2x_1^2$, respectively. On the other hand, the semigroups B_2^1 and A_2^1 are inherently non-finitely based since all Zimin words are isoterms [81, Lemma 3.7]. It follows that a finite semigroup S is inherently non-finitely based if the variety $\operatorname{var}\{S\}$ contains either B_2^1 or A_2^1. Observe that the condition in Theorem 4.5(ii) can hold in a trivial way, namely, when eae or $ea^{d+1}e$ does not belong to M_e, so that both elements do not belong to the same coset of M_e. This is the case for B_2^1; see, for example, Volkov and Gol'berg [102, observation after Proposition 1].

For certain finite monoids M, the condition $B_2^1 \in \operatorname{var}\{M\}$ is not only sufficient but also necessary for M to be inherently non-finitely based.

Lemma 4.6. *Let M be any finite monoid that satisfies the identity $x^{2n} \approx x^n$ for some $n \geq 2$. Suppose that M satisfies at least one of the following four conditions: $|M| \leq 55$, M is regular, the idempotents of M form a submonoid, and all subgroups of M are nilpotent. Then the following conditions are equivalent:*

(a) *M is inherently non-finitely based;*
(b) $B_2^1 \in \operatorname{var}\{M\}$;
(c) *M violates the identity*

$$\big((xy)^n(yx)^n(xy)^n\big)^n \approx (xy)^n. \tag{4.1}$$

Proof. (a) ⇔ (b): This holds by Jackson [41, Theorems 1.4 and 2.2] and Sapir [80, Theorem 2].

(c) ⇒ (b): If M violates the identity (4.1), then $B_2 \in \operatorname{var}\{M\}$ by Sapir and Suhanov [84, Theorem 1], so $B_2^1 \in \operatorname{var}\{M\}$ by Jackson [43, Lemma 1.1].

(b) ⇒ (c): The semigroup B_2^1 violates the identity (4.1) under the substitution $(x, y) \mapsto (a, b)$. Hence, the negation of (c) implies the negation of (b). □

The companion website [87] uses the following procedure to decide whether a finite semigroup S is inherently non-finitely based. Suppose that $e_1, e_2, \ldots, e_r$ are all the idempotents of S. Then by Theorem 4.5(i), it suffices to check if some submonoid $M_i = e_iSe_i$ of S is inherently non-finitely based; this can be achieved by applying Theorem 4.5(ii). As this is the most general result, the website can handle semigroups of order higher than 55; if the semigroup is inherently non-finitely based, then the website would provide its upper hypercenter. The website also allows the user to check if a semigroup is inherently non-finitely based with Lemma 4.6. Results on isoterms are computationally demanding and hence not used by us.

There is yet another method to check if a finite monoid is inherently non-finitely based. For each $n \geq 2$, define the words $[x, y]_1^n, [x, y]_2^n, [x, y]_3^n, \ldots$ over $\{x, y\}$ inductively by $[x, y]_1^n = x^{n-1}y^{n-1}xy$ and $[x, y]_{k+1}^n = \big[[x, y]_k^n, y\big]_1^n$ for each $k \geq 1$. Then for any

Table 6
The semigroups U_7, V_7, and W_7.

U_7	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	1	1	1	2	2	3
3	1	2	3	1	1	3	1
4	4	4	4	4	4	4	4
5	4	4	4	4	5	5	7
6	1	2	3	4	5	6	7
7	4	5	7	4	4	7	4

V_7	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	1	1	1	2	2	3
3	1	2	3	1	2	3	3
4	4	4	4	4	4	4	4
5	4	4	4	4	5	5	7
6	1	2	3	4	5	6	7
7	4	5	7	4	5	7	7

W_7	1	2	3	4	5	6	7
1	1	1	1	1	5	5	5
2	1	2	1	2	5	5	7
3	1	1	3	3	5	6	5
4	1	2	3	4	5	6	7
5	5	5	5	5	1	1	1
6	5	6	5	6	1	1	3
7	5	5	7	7	1	2	1

Table 7
The semigroups U_8 and V_8.

U_8	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	1	1	1	2	2	3	4
3	1	2	3	4	3	4	4	4
4	4	4	4	4	4	4	4	4
5	1	2	3	4	5	6	7	8
6	4	4	4	4	6	6	7	8
7	4	6	7	8	7	8	8	8
8	8	8	8	8	8	8	8	8

V_8	1	2	3	4	5	6	7	8
1	1	1	1	1	5	5	7	7
2	1	2	1	2	5	5	7	8
3	1	1	3	3	5	6	7	7
4	1	2	3	4	5	6	7	8
5	5	5	5	5	7	7	1	1
6	5	6	5	6	7	7	1	3
7	7	7	7	7	1	1	5	5
8	7	7	8	8	1	2	5	5

variety $\mathbf{V}$ generated by a finite semigroup that satisfies the identity $x^{2n} \approx x^n$, the subsequence $\{[x,y]^n_{k!}\}$ converges in the $\mathbf{V}$-free semigroup over $\{x,y\}$; let $[x,y]^n_\infty$ denote the limit of this subsequence [100, Section 4.4].

Lemma 4.7 *(Volkov [100, Proposition 4.4]). Let M be any finite monoid that satisfies the identity $x^{2n} \approx x^n$ for some $n \geq 2$. Then M is inherently non-finitely based if and only if it violates either* (4.1) *or*

$$[\mathbf{e}z\mathbf{e}, (\mathbf{e}y\mathbf{e})^{n-1}\mathbf{e}y^{n+1}\mathbf{e}]^n_\infty \approx \mathbf{e},$$

where $\mathbf{e} = (xyzt)^n$.

The GAP package Smallsemi contains all semigroups of order at most 8 and hence we could routinely run the algorithm outlined after Lemma 4.7 to describe all inherently non-finitely based semigroups of order at most 8. The description involves the following semigroups:

- the minimal inherently non-finitely based semigroups A_2^1 and B_2^1;
- the semigroups U_7, V_7, and W_7 of order 7 given in Table 6;
- the semigroups U_8 and V_8 of order 8 given in Table 7.

The semigroups U_7, W_7, and V_8 generate self-dual varieties while the semigroups V_7 and U_8 do not. Since these five semigroups are monoids, it is routinely checked by Lemma 4.6 that they are all inherently non-finitely based.

Theorem 4.8. *Let S be any semigroup of order at most 8.*

(i) *If $|S| \leq 6$, then S is inherently non-finitely based if and only if it is isomorphic to either A_2^1 or B_2^1.*
(ii) *If $|S| = 7$, then S is inherently non-finitely based if and only if either S is isomorphic to some semigroup from $\{U_7, V_7, \overleftarrow{V}_7, W_7\}$ or S embeds some semigroup from $\{A_2^1, B_2^1\}$.*
(iii) *If $|S| = 8$, then S is inherently non-finitely based if and only if either S is isomorphic to some semigroup from $\{U_8, \overleftarrow{U}_8, V_8\}$ or S embeds some semigroup from $\{A_2^1, B_2^1, U_7, V_7, \overleftarrow{V}_7, W_7\}$.*

We refer to the surveys by Volkov [100,101] for more information on inherently non-finitely based semigroups and the finite basis problem for finite semigroups in general.

5. Open problems

In this section, we pose a number of problems that are naturally prompted by results reported in this survey.

5.1. Varieties with extreme number of subvarieties

Recall that a variety is *small* if it contains only finitely many subvarieties. The variety generated by any finite group is Cross and therefore small [75]. But besides the sufficient condition of Malyshev [67] (Proposition 2.1), very little is known about small varieties generated by a finite semigroup that is not a group.

Problem 5.1. Characterize small varieties generated by a finite semigroup.

The more general problem of investigating all small varieties of semigroups was first suggested by Evans [20, page 38] over 50 years ago. But this problem is infeasible because there exist continuum many varieties of groups with precisely three subvarieties [50].

Example 5.2 *(Sapir [82]).* The class of small varieties of semigroups is closed under neither joins nor covers. More specifically,

(i) there exist finitely generated small varieties $\mathbf{V}_1$ and $\mathbf{V}_2$ whose join $\mathbf{V}_1 \vee \mathbf{V}_2$ contains countably infinitely many subvarieties;
(ii) there exists a finitely generated small variety covered by a finitely generated variety with countably infinitely many subvarieties.

In Example 5.2(i), it is possible for the join $\mathbf{V}_1 \vee \mathbf{V}_2$ to contain continuum many subvarieties if either $\mathbf{V}_1$ or $\mathbf{V}_2$ is allowed to contain countably infinitely many subvarieties [40]. This naturally leads to the following problem.

Table 8
The semigroups E_1, E_2, and E_3.

E_1	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	1	1	2	3
4	1	2	3	4	3	4
5	1	1	1	1	2	5
6	1	2	3	4	5	6

E_2	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	2	1	4	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	1	2	3	4	5	6

E_3	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	2
3	1	1	2	2	3	3
4	1	1	2	2	4	4
5	1	2	3	4	5	6
6	1	2	4	3	6	5

Problem 5.3 *(Jackson [40, Question 3.15]).* Do small varieties of semigroups $\mathbf{V}_1$ and $\mathbf{V}_2$ exist whose join $\mathbf{V}_1 \vee \mathbf{V}_2$ contains continuum many subvarieties?

Problem 5.3 has recently been positively solved for varieties of other related algebras such as monoids [28,44], involution semigroups [56], and involution monoids [57]. In fact, for varieties of monoids, Gusev [28] exhibited two Cross varieties $\mathbb{M}_1$ and $\mathbb{M}_2$ whose join $\mathbb{M}_1 \vee \mathbb{M}_2$ does not only contain continuum many subvarieties but also covers $\mathbb{M}_1$. This example, together with Example 5.2(ii), motivates the following problem.

Problem 5.4. Does a small variety of semigroups exist that is covered by a variety with continuum many subvarieties?

For an overview of results on the lattice of varieties of semigroups, the lattice of varieties of monoids, and the lattice of varieties of involution semigroups, see Shevrin et al. [90], Gusev et al. [31], and Crvenković and Dolinka [13], respectively.

The smallest semigroups currently known to generate varieties with continuum many subvarieties are of order 6, and five examples have so far been found: the inherently non-finitely based semigroups B_2^1 and A_2^1 [40] and the finitely based semigroups E_1, E_2, and E_2 [18,59] in Table 8. As for the other two non-finitely based semigroups of order 6—A_2^g and L_3—the variety $\operatorname{var}\{A_2^g\}$ contains countably infinitely many subvarieties [63], but it is only known that $\operatorname{var}\{L_3\}$ contains at least infinitely many subvarieties.

Problem 5.5. Determine if the variety $\operatorname{var}\{L_3\}$ contains continuum many subvarieties.

It is natural to ask if a semigroup of order less than 6 can generate a variety with continuum many subvarieties. As shown in the supplementary material (Appendix A), every semigroup of order at most 4 generates a variety with at most countably many subvarieties. This is in fact a consequence of a more general result: every variety generated by a semigroup of order at most 4 is *hereditarily finitely based* [51] in the sense that all its subvarieties are finitely based. Since only countably many finite sets of identities exist up to renaming of variables, a hereditarily finitely based variety of algebras with finitely many operations, such as semigroups and monoids, can contain at most countably many subvarieties.

Regarding semigroups of order 5, the monoid P_2^1 obtained from

$$P_2 = \langle a, b \,|\, a^2 = ab = a,\, b^2a = b^2 \rangle = \{a, b, ba, b^2\}$$

plays a crucial role: every semigroup of order 5 that is not equivalent to P_2^1 generates a hereditarily finitely based variety, and it remains open whether or not the variety $\mathrm{var}\{P_2^1\}$ is hereditarily finitely based [53]. Therefore, depending on the solution of the following problem, the semigroup P_2^1 may turn out to be special.

Problem 5.6. Determine which of the following mutually exclusive outcomes is true:

(a) $\mathrm{var}\{P_2^1\}$ contains continuum many subvarieties;
(b) $\mathrm{var}\{P_2^1\}$ contains only countably infinitely many subvarieties but is not hereditarily finitely based;
(c) $\mathrm{var}\{P_2^1\}$ is hereditarily finitely based.

The most striking outcome would be (a): up to equivalence, P_2^1 would be the unique smallest semigroup to generate a variety with continuum many subvarieties. Outcome (b) is also interesting because up to equivalence, P_2^1 would be the unique smallest semigroup that is not hereditarily finitely based. If outcome (c) is true, then every variety generated by a semigroup of order 5 or less would be hereditarily finitely based and so would contain at most countably many subvarieties. Note that if either (b) or (c) is true, then the smallest semigroups that generate a variety with continuum many subvarieties would be of order 6.

Remark 5.7. Recently, Gusev et al. [32] have shown that the variety of monoids generated by P_2^1 is hereditarily finitely based. Consequently, every variety of monoids generated by a monoid of order at most 5 is hereditarily finitely based.

5.2. Finitely universal varieties

Recall that a variety is *finitely universal* if its lattice of subvarieties embeds every finite lattice. Volkov [99] proved that the variety

$$\mathbf{H} = [x^2 \approx yxy]$$

is finitely universal. As observed by Gusev and Lee [30], every known example of finitely generated finitely universal variety contains $\mathbf{H}$.

Problem 5.8. Is there a finite semigroup S such that the variety $\mathrm{var}\{S\}$ is finitely universal but does not contain $\mathbf{H}$?

Every known example of a finitely generated variety with continuum many subvarieties is also finitely universal. It is thus of interest to find a counterexample.

Problem 5.9. Is there a finite semigroup S such that the variety $\mathrm{var}\{S\}$ contains continuum many subvarieties but is not finitely universal?

A finitely generated finitely universal variety can be decomposed as the join of two varieties that are not finitely universal. This is illustrated by the monoids LZ_2^1 and N_2^1, where LZ_2 is the left zero band of order 2 and N_2 is the null semigroup of order 2: the varieties $\mathrm{var}\{LZ_2^1\}$ and $\mathrm{var}\{N_2^1\}$ are not finitely universal but their join is finitely universal [30, Section 6.3]. Since $\mathrm{var}\{LZ_2^1\}$ is small while $\mathrm{var}\{N_2^1\}$ is not [20], the following problem is of fundamental importance.

Problem 5.10. Do small varieties of semigroups $\mathbf{V}_1$ and $\mathbf{V}_2$ exist whose join $\mathbf{V}_1 \vee \mathbf{V}_2$ is finitely universal?

Remark 5.11. There exist two Cross varieties of monoids whose join is finitely universal and has continuum many subvarieties [29].

Problem 5.12. Characterize finitely universal varieties generated by a finite semigroup.

5.3. Identity bases

Recall from Section 4.2 that, up to equivalence, the semigroups B_2^1, A_2^1, A_2^g, and L_3 of order 6 are precisely all minimal non-finitely based semigroups of order 6, and explicit identity bases have been found for the latter two semigroups. Doing the same for B_2^1 and A_2^1 seems extremely challenging.

Problem 5.13. Find explicit identity bases for the semigroups B_2^1 and A_2^1.

The problem of deciding if an identity holds in B_2^1 is co-NP-complete [47,85], while the same problem for A_2^1 is polynomial [47,86]. Therefore, it may be easier to find an explicit identity basis for A_2^1 than for B_2^1.

An identity basis Σ for an algebra A is *irredundant* if every proper subset of Σ fails to be an identity basis for A. Every finitely based algebra has an irredundant identity basis—remove redundant identities from any finite identity basis, one by one, until no redundancies exist. It seemed plausible that any finite semigroup without finite identity bases had an irredundant one. But this optimism was refuted by subsequent examples of finite semigroups without irredundant identity bases, with the non-finitely based semigroup L_3 of order 6 being a smallest possible example; see Lee [54] and the references therein.

On the other hand, it was unknown if a finite semigroup can have an infinite irredundant identity basis, and the existence of such a semigroup has been questioned since the 1970s; see Shevrin and Volkov [91, Question 8.6] and Volkov [101, Problem 2.6]. This question remained open until 2005, when Jackson [42] published a few examples, the smallest of which is of order 9.

Problem 5.14. Is there a semigroup of order at most 8 with an infinite irredundant identity basis?

Remark 5.15. There exists an involution semigroup of order 8 that has an infinite irredundant identity basis [55]. But counterintuitively, when considered as a semigroup, it generates the variety var$\{L_3 \times \mathbb{Z}_3\}$, which has no irredundant identity bases [54].

The finite basis problem has been solved for all semigroups of order at most 6 [60]. One of the most important outcomes of this investigation is the discovery of L_3, a non-finitely based semigroup that has stimulated research in several directions; see Lee [58, Chapter 1].

Problem 5.16. Investigate the finite basis problem for semigroups of order 7.

Apart from completely solving this problem, it would be very interesting to discover new examples of non-finitely based semigroups S that are essentially "different" from the four minimal non-finitely based semigroups B_2^1, A_2^1, A_2^g, and L_3, for instance, S is not constructed from any of them.

5.4. Number of varieties

The number of varieties generated by a semigroup of order at most 5 is 218. The number of varieties with a primitive generator of order 6 has not yet been determined but is known to be at least 463; see Table 4. Among these 463 varieties, 49 are known (45 finitely based and 4 non-finitely based). Therefore, the following problem consists of at least 414 different cases.

Problem 5.17. Identify all varieties with a primitive generator of order 6. Find a Bas-Max system for each of these varieties.

In the companion website [87], the tab *Conjectures* contains 463 conjectures, each one a problem by itself. We expect to include a similar file for semigroups of order 7. There are 73,807 non-isomorphic semigroups of order 7 whose varieties do not coincide with known varieties stored in our database.

Problem 5.18. Let $\nu(n)$ denote the number of varieties with a primitive generator of order n.

(i) Find good upper and lower bounds for $\nu(n)$.
(ii) Find a closed formula or a recursive function for $\nu(n)$.

It follows from Table 4 that $\nu(1) = 1$, $\nu(2) = 5$, $\nu(3) = 14$, $\nu(4) = 53$, $\nu(5) = 145$, and $\nu(6) \geq 463$.

5.5. Groups

Problem 5.19. Given a finite group G, find good bounds for

(i) the number of critical groups in $\mathrm{var}\{G\}$;
(ii) the order of the largest critical group in $\mathrm{var}\{G\}$;
(iii) the number of subvarieties of $\mathrm{var}\{G\}$;
(iv) the number of varieties covered by $\mathrm{var}\{G\}$.

Solve the same problems for the class $\mathbf{C}(e, m, c)$ introduced in Section 3.7.

Identity bases for the varieties generated by dihedral groups are known. However, the varieties covered by them are not completely known; see (iv) on page 702. The dihedral groups D_n and D_{2n} with odd n belong to $\mathbf{A}_n\mathbf{A}_2$. Therefore, Houghton's theorem (see page 714) solves the problem of finding the subvarieties. But the problem is open for the other cases.

Problem 5.20. Find all the varieties covered by a variety generated by a dihedral group and provide an identity basis for each.

The smallest undecided case in Problem 5.20 is the dihedral group D_8.

Identity bases are known for many other groups [11,12,93]. Therefore, the previous problem applies to them as well.

Problem 5.21. Let G be a finite group for which an identity basis is known. Find all the varieties covered by $\mathrm{var}\{G\}$ and provide an identity basis for each.

5.6. Computational questions

Recall that a description of all inherently non-finitely based semigroups of order at most 8 is given in Theorem 4.8.

Problem 5.22. Describe all inherently non-finitely based semigroups of order n, for each $n \geq 9$.

The website contains implementations of algorithms that answer questions about varieties of semigroups or groups, but it would be interesting to investigate the algorithms from a complexity point of view.

Problem 5.23. Study the complexity of the algorithmic problems of this paper and other related problems.

Data availability

No data was used for the research described in the article.

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Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://ars.els-cdn.com/content/image/1-s2.0-S0021869323003356-mmc1.pdf.

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