Self-inductance and Magnetic Flux

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The canonical equation for self-inductance involving magnetic flux is examined, and a more general form is presented that can be applied to continuous current distributions. We attempt to clarify and extend the use of the standard equation by recasting it in its more versatile form.

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I. INTRODUCTION

Self-inductance is typically calculated in reference to the *total* current I of some circuital configuration and to the magnetic flux Φ through a relevant surface that the current bounds

$$L = \frac{\Phi}{I}. \tag{1}$$

Generally, the current may be filamentary (i.e., it follows a singular contour) or distributed over a surface or in a volume. The magnetic flux through an area $d\vec{\sigma}$,

$$d\Phi = \vec{B}(\vec{r}) \cdot d\vec{\sigma}, \tag{2}$$

involves the magnetic field that is created by the current distribution. The simplest, *in principle*, application of Eq. (1) is that of a single circular current loop (see Appendix A). In cases in which the current is not filamentary, but rather is distributed across a surface or within a volume, the surface of integration associated with the application of Eq. (1) becomes unclear, especially to students first introduced to magnetostatics. The simplest example of this case is the solenoid, but it is nearly always treated as a filamentary current bounding an effective area so that Eq. (1) is readily applicable (see Sec. III). In David Griffiths's textbook, *Introduction to Electrodynamics*, he states that in cases in which current is not confined to a single path, "it can be very *tricky* to get the inductance" from Eq. (1).[1] General physics textbooks never address this matter (for example, Refs. 2–4), nor do other undergraduate textbooks (for example, Refs. 5–7). Even advanced physics textbooks on electrodynamics are silent on the matter (for example, Ref. 8). It is curious that a more versatile version of Eq. (1) exists but is never presented in any of the references cited.

In the pedagogical literature on inductance, one finds references to "flux linkage" or "flux weighting" but no elaboration on these (see, for example, Refs. 9 and 10, and Ref. 11). The idea is assumed to be common knowledge and appears to be familiar to the electrical engineering community. We shall see that the more versatile version of Eq. (1), derived in Sec. II, will show that Φ is indeed a weighted flux. This version of the equation deserves to be explained, which this paper attempts to do, thereby filling what appears to be a small void in the physics literature. This version should clarify the approach to *any* self-inductance problem encountered by students exposed to calculus-level electromagnetism and intent on using the flux method.

Let us begin by pointing out that parts, or segments, of a complete circuit, such as the wire we will consider in Sec. II, have well-defined *partial* inductances despite there not being an explicit current loop. Given any current loop, let $\vec{A}$ be the magnetic vector potential so that

$$\vec{B} = \nabla \times \vec{A}. \tag{3}$$

The flux in Eq. (1) can, therefore, be expressed as

$$\Phi = \oint \vec{A} \cdot d\vec{\ell}, \tag{4}$$

and one can then define the partial inductance of any segment, for example, the straight (filamentary) segment depicted in Fig. 1, to be

$$L_{partial} = \frac{1}{I}\int \vec{A}' \cdot d\vec{\ell}, \tag{5}$$

where the integral is taken along the length of the segment and $\vec{A}'$ is the vector potential of *only* that segment. Note that the sum of the partial inductances is *not* the loop inductance as there are partial mutual inductances (between that segment and other parts of the current loop) that contribute to this. If we wish to return to Eq. (1) and calculate the partial inductance via a flux through a loop, we can create an integration path that includes the segment, as long as the line integral over the additional path is zero. For example, given an (albeit unphysical) isolated linear conductor carrying a current I, the partial inductance can be calculated by integrating $\vec{A}' \cdot d\vec{\ell}$ around the clockwise rectangular path depicted in the figure. This closed path is taken to extend to infinity on the right so that $A' \to 0$ on the right leg of the path. Also, the vector potential is perpendicular to the horizontal (dashed) segments in the figure so $\vec{A}' \cdot d\vec{\ell} = 0$ on the top and bottom legs. Application of Eq. (1) to this area will, therefore, yield $L_{partial}$ for the linear conductor. A more detailed exposition of partial inductance can be found in Ref. 12.

When calculating a flux in cases of distributed currents, such as the linear conductor considered in Sec. II, there will be a part of the flux area that overlaps the conductor and a part that does not. The inductance calculation can then be split into the contribution from the overlap, referred to as the internal inductance, and that from the non-overlap region, called the external inductance.

In Sec. II, we will derive the versatile version of Eq. (1) and apply it to calculate the self-inductances of a straight

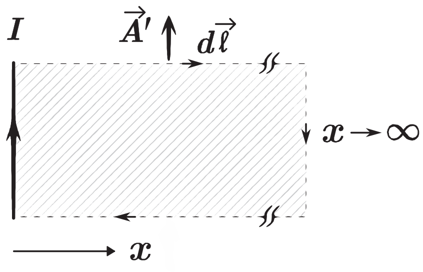

Fig. 1. Isolated linear segment of a current loop with integration contour for partial inductance.

wire, of a solenoid (Sec. III), of a ribbon (Sec. IV), and of an annular current (Sec. V).

II. GENERAL EQUATION FOR SELF-INDUCTANCE

To derive the more general form of Eq. (1), we start with the magnetic field energy

$$\frac{1}{2\mu_0}\int\int\int_{\alpha\sigma}\vec{B}^2 d\tau = \frac{1}{2}LI^2, \tag{6}$$

where $d\tau$ is the volume element, $\alpha\sigma$ is shorthand for "all space," and I is the total current of the circuital configuration that creates the magnetic field $\vec{B}$. Equation (6) could be solved to find the self-inductance of the circuit. The basis for the following derivation can be found, for example, in Ref. 1. If one considers a spherical surface with a radius $r \to \infty$ enclosing the current distribution, the following surface integral ($d\vec{\sigma}$ is the surface element) vanishes in this limit, since $A \to 1/r$, $B \to 1/r^2$, and the surface area increases as r^2,

$$\oint\!\!\oint_{\partial V}(\vec{A}\times\vec{B})\cdot d\vec{\sigma} \to 0, \tag{7}$$

where ∂V is the (closed) boundary of the volume. By Gauss's theorem,

$$\oint\!\!\oint(\vec{A}\times\vec{B})\cdot d\vec{\sigma} = \int\int\int\nabla\cdot(\vec{A}\times\vec{B})d\tau. \tag{8}$$

As long as the integration is over all space, Eq. (6) can be written as

$$L = \frac{1}{\mu_0 I^2}\int\int\int_{\alpha\sigma}\left[\vec{B}^2 - \nabla\cdot(\vec{A}\times\vec{B})\right]d\tau. \tag{9}$$

Mathematically, using Ampere's law

$$\vec{B}^2 - \nabla\cdot(\vec{A}\times\vec{B}) = \vec{A}\cdot(\nabla\times\vec{B}) = \mu_0\vec{A}\cdot\vec{J}. \tag{10}$$

Consequently,

$$L = \frac{1}{I^2}\int\int\int_{\alpha\sigma}\vec{A}\cdot\vec{J}\,d\tau. \tag{11}$$

The volume current density can be expressed as filamentary differential sub-currents,

$$\vec{J}(\vec{r})\,d\tau = \hat{n}\,J\,da_\perp d\ell = dI d\vec{\ell}, \tag{12}$$

where $\hat{n}$ is the unit vector in the direction of current flow and $da_\perp$ is the area perpendicular to $\hat{n}$ of the volume element $d\tau$ whose length is $d\ell$ so that Eq. (11) becomes

$$L = \frac{1}{I^2}\int dI\oint_{\partial\Sigma}\vec{A}\cdot d\vec{\ell}, \tag{13}$$

$$= \frac{1}{I^2}\int dI\int\int_\Sigma\nabla\times\vec{A}\cdot d\vec{\sigma}, \tag{14}$$

$$= \frac{1}{I^2}\int dI\int\int_\Sigma\vec{B}\cdot d\vec{\sigma}, \tag{15}$$

where Eq. (14) follows from Eq. (13) by Stokes's theorem. Note that $\partial\Sigma$ is the (closed) boundary contour for dI; the current element dI flows around this contour $\partial\Sigma$, which bounds the flux surface Σ (Fig. 2). So if we define

$$\Phi_\Sigma = \int_\Sigma d\Phi, \tag{16}$$

as the flux linked with this surface Σ, then the correct flux in Eq. (1) is, in fact, the weighted integral

$$\Phi = \int\frac{dI}{I}\,\Phi_\Sigma, \tag{17}$$

so that

$$L = \frac{1}{I^2}\int dI\,\Phi_\Sigma. \tag{18}$$

For cases in which the current is confined to a single filamentary path, Eq. (18) reduces to its standard form, Eq. (1).

To illustrate a naive use of Eq. (1) that results in an incorrect result, let us consider the case of a straight cylindrical conductor of radius R and length ℓ. We assume that the current is uniformly distributed over the circular cross section. From Ampere's law, the magnetic field inside the cylinder at a distance r from the central axis is

$$\vec{B}(r) = \frac{\mu_0}{2\pi}\frac{Ir}{R^2}\hat{\phi}, \tag{19}$$

where $\hat{\phi}$ is the azimuthal unit vector in cylindrical coordinates. The student presented with the problem of finding the internal magnetic field's contribution to the inductance would likely identify the rectangular area, $\mathcal{ABCD}$ (shown in Fig. 3(a) with a representative gray differential strip $d\sigma$), as the relevant *total* area over which the magnetic flux should be calculated, which is correct. Indeed, Griffiths points out that the return current can be assumed to follow along the surface (i.e., from $\mathcal{B}$ to $\mathcal{C}$ in the figure) just beyond a thin

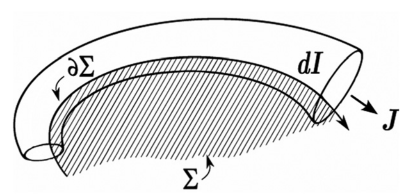

Fig. 2. A portion of a filamentary current's path and associated bounded surface depicted within a tubular volume current distribution.

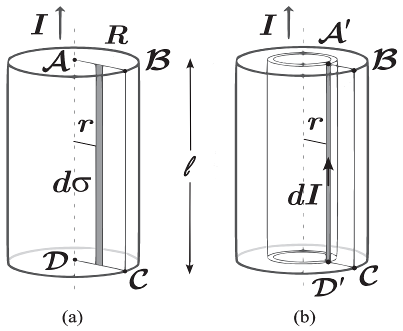

Fig. 3. (a) A long straight conductor of radius R, length ℓ, and through which flows a current I. (b) The same conductor with representative current element dI identified.

insulating sheath.[13] However, a straightforward (albeit naive) calculation using Eq. (1) yields $L_{int} = \mu_0 \ell / 4\pi$, which is wrong by a factor of two.

This problem is solved correctly in Ref. 11. Rosa calculates separately the contributions to the self-inductance due to the internal field of the wire and to the external field using magnetic fluxes. To determine the internal contribution (which is specifically the part we are considering here), he calculates a weighted flux (without explanation), which yields the correct answer. He confirms the result by comparing it with that coming from Eq. (6). Griffiths explicitly espouses this energy method when a current is not confined to a single path. However, the volume integral in Eq. (6) may be intractable in many cases, such as in the case of a circular current loop, so that it is useful to have a version of Eq. (1) that removes any confusion when calculating fluxes in the presence of surface or volume current distributions. We point out that implicit in Griffiths (by using Eq. (11)) is also the option to work with the more primal vector potential $\vec{A}$ rather than the energy or the flux.

Let us now reconsider the self-inductance of the wire due to the internal field using Eq. (18). Figure 3(b) shows a representative filamentary current element $dI(r, \phi)$,

$$dI(r, \phi) = \frac{I}{\pi R^2} r dr d\phi. \tag{20}$$

As a singular (linear) current, the relevant area of integration is the loop, $\mathcal{A}'\mathcal{BCD}'$ (i.e., a rectangle within the conductor with the current as one side). This follows from the discussion on partial internal inductance in the Introduction. The flux through this area is

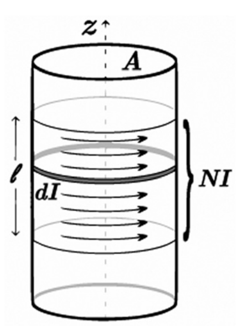

Fig. 4. Section of an infinite solenoid.

$$\Phi_\Sigma(r, \phi) = \int_r^R B(r')(\ell dr'), \tag{21}$$

$$= \frac{\mu_0 \ell I}{4\pi R^2}(R^2 - r^2). \tag{22}$$

Consequently, the self-inductance contribution due to the field within the conductor follows correctly from Eq. (18):

$$L_{int} = \frac{1}{I^2} \int \Phi_\Sigma(r, \phi) dI(r, \phi), \tag{23}$$

$$= \frac{\mu_0 \ell}{8\pi}. \tag{24}$$

Although the calculation of the self-inductance contribution due to the external field does not present any problems using Eq. (1) since its determination involves, in effect, a filamentary current, we include it for completeness in Appendix B.

III. SELF-INDUCTANCE OF A SOLENOID

We now consider the simple case of the solenoid (Fig. 4). The solenoid is the most common example of the use of Eq. (1) in calculating self-inductance. Students generally have no difficulty in determining that the cross section A of the solenoid is the surface through which the flux has to be calculated. However, they are often confused by how this surface changes with the finite length ℓ or the number of windings N. In fact, the total area per axial length ℓ is that of N disks, or that of a spiral surface of N levels so that the total surface to be considered for the flux calculation is $A_{tot} = NA$. If the solenoid were infinite, the magnetic field is then constant ($B = \mu_0 NI/\ell$), the flux is simply $\Phi = BA_{tot}$, and the self-inductance per unit length $NBA/I\ell = \mu_0 (N/\ell)^2 A$.

The calculation is also easily handled with Eq. (18) by denoting the effective surface current density $\mathbb{K} = NI/\ell$. Taking the axial coordinate to be z, $dI = \mathbb{K} dz$. The flux contour is a circle whose disk surface area is A and the associated flux $\Phi_\Sigma = BA$. The self-inductance for a solenoid of length ℓ is

$$L = \frac{1}{I^2} \int dI \, \Phi_\Sigma = \frac{\mathbb{K} BA}{I^2} \int dz = \frac{NBA}{I} = \frac{\mu_0 N^2 A}{\ell}, \tag{25}$$

where the final equality assumes that the magnetic field is the same as for the infinite solenoid.

IV. SELF-INDUCTANCE OF A RIBBON

We now illustrate a case in which the flux method is better suited than the energy method to determine an inductance. Consider a long ribbon of length ℓ and width $w \ll \ell$ with uniform surface current density $\mathbb{K} = I/w$. We seek the internal inductance, i.e., that part of the partial inductance due solely to the magnetic field within the conductor (as in the case of the linear conductor in Sec. II). This inductance component involves no volume, so the energy method would not be obviously viable. One proceeds with the flux method by determining the magnetic field along s, the direction perpendicular to the ribbon length, with $s = 0$ at one edge (the left side of the ribbon in Fig. 5). Since the differential current elements $dI = \mathbb{K} ds$ are singular straight currents with well-known magnetic field, an application of the superposition principle yields

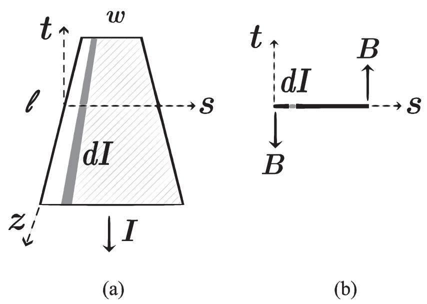

Fig. 5. (a) A surface current forming a long ribbon of width w. (b) A head-on view.

$$B(s)=\begin{cases}-\dfrac{\mu_0\mathbb{K}}{2\pi}\ln\left(\dfrac{w-s}{s+\varepsilon}\right), & 0\le s\le w/2,\\[2ex] +\dfrac{\mu_0\mathbb{K}}{2\pi}\ln\left(\dfrac{s}{w-s+\varepsilon}\right), & w/2\le s\le w,\end{cases} \tag{26}$$

where ε regularizes the infinity at the edges. There is obviously a symmetry about the central line of the ribbon. Each current element is part of an infinite rectangular contour whose area is hatched in the figure for a give dI (in gray). The contour extends beyond the ribbon (i.e., $s>w$; recall again the discussion on partial and internal/external inductance in the Introduction), but we are only considering the internal inductance now. The contour can be extended either to the right (as we are doing) or equivalently to the left. Next we calculate the flux for this area

$$\Phi_\Sigma(s)=\ell\int_{w-s}^{w}B(s')ds',\quad s\le w/2, \tag{27}$$

$$=\frac{\mu_0\mathbb{K}\ell}{2\pi}\left[w\ln w-s\ln s-(w-s)\ln|w-s|\right]. \tag{28}$$

Because of the symmetry $\Phi_\Sigma(s)=\Phi_\Sigma(w-s)$ for $s\ge w/2$. Now using Eq. (18), we get

$$L_{int}=\frac{1}{I^2}\int_0^w dI\,\Phi_\Sigma(s), \tag{29}$$

$$=\frac{2\mathbb{K}}{I^2}\int_0^{w/2}\Phi_\Sigma(s)\,ds, \tag{30}$$

$$=\frac{\mu_0\ell}{4\pi}. \tag{31}$$

The external inductance of the ribbon can be calculated using the results of Appendix B. Instead of the filamentary current, we now have a surface current $\mathbb{K}=I/w$. In Eq. (B2), let $I\to dI'=\mathbb{K}ds'$ and $s\to s-s'$ where s' locates the element of current dI' (Fig. 6), then

$$B=\frac{\mu_0\mathbb{K}}{4\pi}\int_{-w/2}^{+w/2}\frac{ds'}{(s-s')}\left[\frac{\ell-z}{\sqrt{(\ell-z)^2+(s-s')^2}}+\frac{z}{\sqrt{z^2+(s-s')^2}}\right]. \tag{32}$$

The rest of the calculation proceeds the same, once the order of integration is changed to

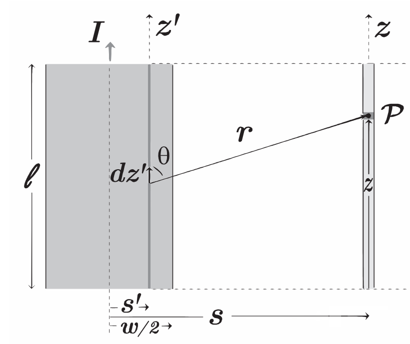

Fig. 6. Magnetic flux external to the ribbon current.

$$\Phi=\int_{-w/2}^{+w/2}ds'\int_{w/2}^{\infty}ds\int_0^{\ell}dz\,B=\frac{\mu_0 I\ell}{2\pi}\ln\frac{2\ell}{w}. \tag{33}$$

The total (partial) inductance of the ribbon is, therefore,

$$L=\frac{\mu_0\ell}{2\pi}\left(\ln\frac{2\ell}{w}+\frac{1}{2}\right). \tag{34}$$

V. THE ANNULUS

As a non-trivial application of Eq. (18), we outline the calculation of the self-inductance of a 2D annulus (or washer). Since an annulus is a flattened hollow cylinder, the idea employed in the solenoid case also applies. Although in the annulus case, the magnetic field resulting from the current distribution will be far more complicated than the constant one of the infinite solenoid. In Fig. 7, a uniform surface current, $\mathbb{K}$, flows in a circular pattern forming an annulus of current with inner radius, a, and outer radius, b. If one takes the annulus to be a flat spiral coil, as in an RFID tag antenna, for example, one can express the surface current density as $\mathbb{K}=NI/(b-a)$. Naively trying to apply Eq. (1), a student may be confused as to which flux Φ should be used. The alternative is Eq. (6), but this would require knowledge of the magnetic field in all space and would involve a volume integral that is certainly daunting, if not altogether unfeasible. As in the ribbon case of Sec. IV, the flux method is likewise preferable, as it requires only the knowledge of the magnetic field in the plane of the annulus (as is the case for the circular current; see Appendix A). Moreover, Eq. (18) removes the ambiguity in approaching this case using the flux method.

In Fig. 7, a representative circular current element within the annulus is identified, of radius r and current $dI=\mathbb{K}dr$. The self-inductance then follows from Eq. (18) as

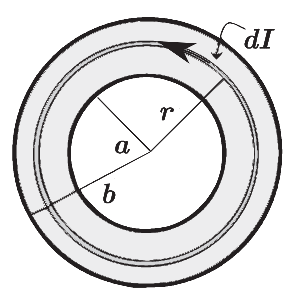

Fig. 7. Uniform surface current distribution forming an annulus.

$$L = \frac{1}{I^2}\int_a^b \Phi_\Sigma(r)(\mathbb{K}dr), \tag{35}$$

where $\Phi_\Sigma(r)$ is the magnetic flux through a disk of radius, r, due to the annular current (see the end of Appendix A for more details). The determination of this flux and its integration in Eq. (35) is now the crux of the matter and beyond the scope of this paper. The work on the circular current loop of Appendix A can serve as a basis for this calculation since it lays out how to calculate the flux contributions from the (singular) sub-currents of the annulus. Although it avoids the complications of the field energy method, the calculation, using the flux method at least, is anything but trivial. Reference 9 presents an alternate method for calculating self-inductance based on the relationship between power and current rather than flux and current, and perhaps represents a more efficient method in this case.

VI. CONCLUSION

Although self-inductance can be calculated via various methods, including using the magnetic field energy or the magnetic vector potential, students in introductory magnetostatics are often asked to derive it via magnetic fluxes. From this perspective, we believe it is well worth presenting Eq. (18), although it is absent from the more common textbooks on electromagnetism. By recasting Eq. (1) in the more explicit, or versatile, form of Eq. (18), instructors will provide students with a more valuable tool in general and an invaluable tool in problems without a filamentary current.

AUTHOR DECLARAIONS

Conflict of Interest

The authors have no conflicts of interest.

APPENDIX A: THE CIRCULAR CURRENT

For completeness, in this appendix, we calculate the self-inductance of a circular current of radius R (Fig. 8), where there is no conceptual problem in using Eq. (1). The magnetic flux of the circular current loop through its bounded area is first calculated. To do so, the magnetic field at a point located at $\vec{r}$, due to the circular current, is calculated using the Biot–Savart law. Without loss of generality, $\vec{r} = r\hat{x}$, then

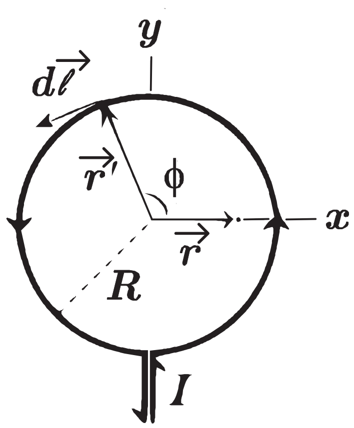

Fig. 8. Circular current.

$\vec{r}' = (\hat{x}\cos\phi + \hat{y}\sin\phi)R$ and $d\vec{\ell} = (-\hat{x}\sin\phi + \hat{y}\cos\phi)Rd\phi$, with ϕ being the angle between $\hat{x}$ and $\vec{r}'$, $\hat{x}$ and $\hat{y}$ the unit vectors along the x and y directions. Consequently,

$$\vec{B}(\vec{r}, R) = \frac{\mu_0}{4\pi}\int \frac{Id\vec{\ell} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}. \tag{A1}$$

The only non-zero component of $\vec{B}$ is along the z direction,

$$B(r, R) = \frac{\mu_0 IR}{2\pi}\int_0^\pi d\phi \frac{R - r\cos\phi}{[r^2 - 2rR\cos\phi + R^2]^{3/2}}, \tag{A2}$$

where $B > 0$ for $r < R$, and $B < 0$ for $r > R$. Let Σ be the disc bounded by the current. Using Eq. (1), it is straightforward to calculate the self-inductance in this case

$$L = \frac{1}{I}\int_\Sigma \vec{B}\cdot d\vec{a}, \tag{A3}$$

$$= \frac{1}{I}\int_0^{R_-} B(r, R)(2\pi r dr), \tag{A4}$$

where $R_- = R - \varepsilon$ and $\varepsilon \to 0$ is a regularizer used to circumvent the fact that the magnetic field diverges near the current loop. Replacing $B(r, R)$ by its expression (Eq. (A2)), Eq. (A4) becomes

$$L = \mu_0 R\int_0^\pi d\phi J(\phi), \tag{A5}$$

where

$$J(\phi) = \int_0^{R_-} dr \frac{r(R - r\cos\phi)}{[r^2 - 2rR\cos\phi + R^2]^{3/2}}. \tag{A6}$$

To evaluate this integral, we make the substitution, $z = (r/R - \cos\phi)\cos\phi$,

$$J(\phi) = -\cos\phi\int_{-\cos^2\phi}^{-\cos^2\phi + \eta\cos\phi} dz \frac{(z + \cos^2\phi)(z - \sin^2\phi)}{[z^2 + \sin^2\phi\cos^2\phi]^{3/2}}, \tag{A7}$$

where $\eta = R_-/R$. Then the second substitution, $z = \sin\phi\cos\phi\sinh\alpha$, yields after integration on α,

$$J(\phi) = 1 - \cos\phi\sinh^{-1}(\cot\phi) - \cos\phi\sinh^{-1}\left(\frac{\eta - \cos\phi}{\sin\phi}\right) + \frac{2\eta\cos\phi - 1}{\sqrt{1 - 2\eta\cos\phi + \eta^2}}. \tag{A8}$$

Then the integration with respect to ϕ yields

$$\frac{L}{\mu_0 R} = \int_0^\pi d\phi J(\phi), \tag{A9}$$

$$= \frac{1 + \eta^2}{|1 - \eta|}\int_0^{\pi/2} d\theta \frac{1}{\sqrt{1 + k^2\sin^2\theta}} - |1 - \eta|\int_0^{\pi/2} d\theta\sqrt{1 + k^2\sin^2\theta}, \tag{A10}$$

$$= \frac{1+\eta^2}{|1-\eta|} F\left(\frac{\pi}{2}, ik\right) - |1-\eta| E\left(\frac{\pi}{2}, ik\right), \quad \text{(A11)}$$

where $k^2 = 4\eta/(1-\eta)^2$, and F, E are elliptic integrals of the first and second kinds, respectively. Finally, the result can be expressed in terms of complete elliptic integrals K and E as

$$\frac{L}{\mu_0 R} = \frac{R^2 + R_-^2}{R(R+R_-)} K(\kappa) - \frac{R+R_-}{R} E(\kappa), \quad \text{(A12)}$$

where $\kappa = 4RR_-/(R+R_-)^2 < 1$. For $\kappa < 1$, the elliptic integrals can be approximated by a polynomial[14]

$$K(\kappa) = \ln 4 + a_1\tilde{\kappa} + a_2\tilde{\kappa}^2 + \left(\frac{1}{2} + b_1\tilde{\kappa} + b_2\tilde{\kappa}^2\right) \ln\frac{1}{\tilde{\kappa}}, \quad \text{(A13)}$$

$$E(\kappa) = 1 + c_1\tilde{\kappa} + c_2\tilde{\kappa}^2 + \left(d_1\tilde{\kappa} + d_2\tilde{\kappa}^2\right) \ln\frac{1}{\tilde{\kappa}}, \quad \text{(A14)}$$

where $a_1 = 0.1119723$, $a_2 = 0.0725296$, $b_1 = 0.1213478$, $b_2 = 0.0288729$, $c_1 = 0.4630151$, $c_2 = 0.1077812$, $d_1 = 0.2452727$, $d_2 = 0.0412496$, and where $\tilde{\kappa} = 1 - \kappa = (R - R_-)^2/(R + R_-)^2 \to (\varepsilon/2R)^2$ as $\varepsilon \to 0$. In this limit,

$$L = \mu_0 R\left[\ln\frac{8R}{\varepsilon} - 2\right]. \quad \text{(A15)}$$

Some of the results above can be carried over to the annulus case considered in the main text. In Eq. (A12), replacing R_- with r, one notes that L represents the flux distribution for the annulus

$$L \equiv \frac{d\Phi}{dI}(r;R) = \mu_0\left[\frac{R^2+r^2}{(R+r)} K\left(\frac{4Rr}{(R+r)^2}\right) - (R+r) E\left(\frac{4Rr}{(R+r)^2}\right)\right], \quad \text{(A16)}$$

so that $d\Phi(r;R)$ is the flux through a circular region of radius, r, due to a co-centered circular current of radius, R. Therefore, the flux associated with a differential circular current element of radius r within the annulus of inner radius a, outer radius b, and surface current density $\mathbb{K}$ is

$$\Phi_\Sigma(r) = \int \frac{d\Phi}{dI}(r;R)dI = \int_a^b \frac{d\Phi}{dI}(r;R)(\mathbb{K}dR), \quad \text{(A17)}$$

which is the flux that appears in Eq. (35).

APPENDIX B: THE LINEAR CONDUCTOR'S EXTERNAL FIELD SELF-INDUCTANCE

We complete the case of the wire, or linear conductor, by calculating the (partial) self-inductance contribution due to the external field. We assume the conductor to have a length ℓ and a radius $R \ll \ell$. To calculate the flux, we start by calculating the magnetic field at point $\mathcal{P}$ external to the conductor due to a uniform current I flowing along the wire. The position $\mathcal{P}$ is defined by parameters s and z or by z', r and the angle θ. These quantities are defined in Fig. 9. Since $\mathcal{P}$ lies

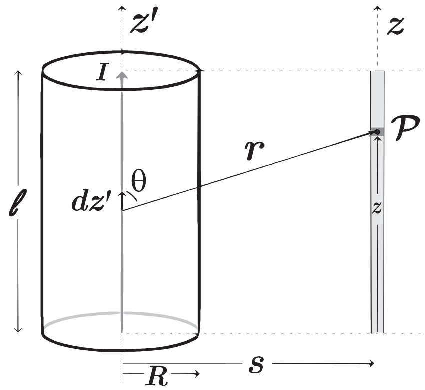

Fig. 9. Magnetic flux external to the linear conductor.

outside of the conductor, the magnetic field created by the wire is identical to the one created by a line current running down the center of the cylinder. Using the Biot–Savart law, the contribution from each $I\,d\vec{z}'$ (where both the z and z' axes run positive upwards in Fig. 9) is azimuthal (i.e., directed into the page at $\mathcal{P}$ in the figure) so

$$B(s,z) = \frac{\mu_0 I}{4\pi}\int_0^\ell \frac{dz' \sin\theta}{r^2}. \quad \text{(B1)}$$

Now note that $z - z' = s\cot\theta$. Then

$$B(s,z) = \frac{\mu_0 I}{4\pi s}\int_{\theta_I}^{\theta_F} \sin\theta d\theta = \frac{\mu_0 I}{4\pi s}[\cos\theta_I - \cos\theta_F]$$
$$= \frac{\mu_0 I}{4\pi s}\left[\frac{\ell - z}{\sqrt{(\ell - z)^2 + s^2}} + \frac{z}{\sqrt{z^2+s^2}}\right]. \quad \text{(B2)}$$

As discussed in the Introduction, the area to consider for the *external* partial inductance in the case of a linear conductor is a rectangular one, in this case, of height ℓ, and extending horizontally from the surface of the conductor to infinity. A representative element of this area is shown in gray in Fig. 9. The differential flux is then integrated over this area to give

$$\Phi = \int_R^\infty ds \int_0^\ell dz\, B = \frac{\mu_0 I}{2\pi}\int_R^\infty \left[\frac{\sqrt{\ell^2+s^2}}{s} - 1\right] ds. \quad \text{(B3)}$$

Letting $s = \ell\tan\phi$ in Eq. (B3) then yields

$$\Phi = \frac{\mu_0 I}{2\pi}\left[\left[\int_{\phi_R}^{\pi/2} \ell(\csc\phi + \sec\phi\tan\phi)d\phi - \int_R^\infty ds\right], \quad \text{(B4)}$$

$$= \frac{\mu_0 I}{2\pi}\left[-\ell\ln\left|\frac{\ell+\sqrt{\ell^2+s^2}}{s}\right| + \sqrt{\ell^2+s^2} - s\right]_R^\infty, \quad \text{(B5)}$$

$$= \frac{\mu_0 I}{2\pi}\left[\ell\ln\left|\frac{\ell+\sqrt{\ell^2+R^2}}{R}\right| - \sqrt{\ell^2+R^2} + R\right]. \quad \text{(B6)}$$

Therefore,

$$L_{ext} \approx \frac{\mu_0 \ell}{2\pi}\left[\ln\frac{2\ell}{R} - 1\right]. \quad \text{(B7)}$$

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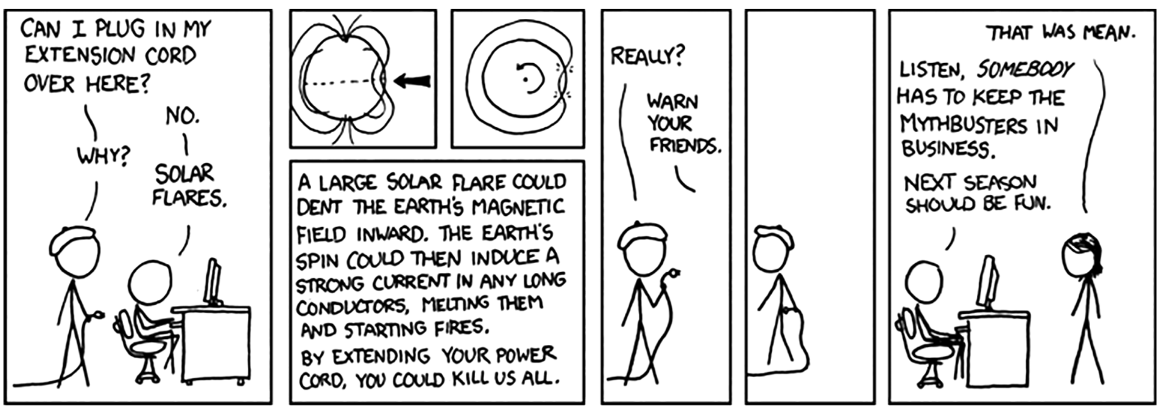

The MythBusters need to tackle whether a black hole from the LHC could REALLY destroy the world. (Source: https://xkcd.com/509)