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A Classification of Hull Operators in Archimedean Lattice-Ordered Groups With Unit

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Abstract. The category, or class of algebras, in the title is denoted by **W**. A hull operator (ho) in **W** is a reflection in the category consisting of **W** objects with only essential embeddings as morphisms. The proper class of all of these is **hoW**. The bounded monocoreflection in **W** is denoted B. We classify the ho's by their interaction with B as follows. A "word" is a function $w : \mathbf{hoW} \longrightarrow \mathbf{W}^{\mathbf{W}}$ obtained as a finite composition of B and x a variable ranging in **hoW**. The set of these,"Word", is in a natural way a partially ordered semigroup of size 6, order isomorphic to F(2), the free $0-1$ distributive lattice on 2 generators. Then, **hoW** is partitioned into 6 disjoint pieces, by equations and inequations in words, and each piece is represented by a characteristic order-preserving quotient of Word ($\approx$ F(2)). Of the 6: 1 is of size ≥ 2, 1 is at least infinite, 2 are each proper classes, and of these 4, all quotients are chains; another 1 is a proper class with unknown quotients; the remaining 1 is not known to be nonempty and its quotients would not be chains.

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1 Introduction

The present work seems to represent new theory imposed on the lattice F(2). This requires some background material from lattice-ordered groups (ℓ-groups) and topology that we review here. For additional information, we refer the reader to [6], [1] and [14] for ℓ-groups, and [15], [18] for topology and $C(X)$.

The following alternative description of a hull operator (in various classes of algebras $\mathbf{C}$ including $\mathbf{W}$) is provided in [11]. A hull operator is an "essential closure operator" (with isomorphic objects identified), that is, $h : \mathbf{C} \longrightarrow \mathbf{C}$ is a function such that for every $C \in \mathbf{C}$ there is $C \overset{\mathrm{e}}{\leq} hC$, where $\leq$ (respectively, $\overset{\mathrm{e}}{\leq}$) signifies embeds as a subobject (respectively, an essential subobject) and for which

(i) $hC = h(hC)$;

(ii) if $C \overset{\mathrm{e}}{\leq} D \leq hC$, then $hD = hC$;

(iii) hC is unique up to a unique isomorphism over C.

As demonstrated in [9], the collection of hull operators on $\mathbf{W}$, $\mathbf{hoW}$, is a complete lattice with a partial ordering defined "pointwise" as follows. For h_1, $h_2 \in \mathbf{hoW}$, $h_1 \leq h_2$ means for every $G \in \mathbf{W}$, $G \overset{\mathrm{e}}{\leq} h_1G \overset{\mathrm{e}}{\leq} h_2G$ (up to unique isomorphism over G). The bottom of the lattice $\mathbf{hoW}$ is the identity operator Id, and the top is the essential completion hull operator e.

The category $\mathbf{W}$ is much informed by the Yosida Representation Theorem, which we now review.

For a Tychonoff space X (usually compact) define

$$D(X) \equiv \{f : X \longrightarrow [-\infty, \infty] \,|\, f \text{ is continuous and } f^{-1}(\mathbb{R}) \text{ is dense in } X\}.$$

Note, $D(X)$ with the pointwise order and addition is a lattice but is usually not a group (as addition is not always fully defined). We now recall the Yosida Representation Theorem from [23].

Theorem 1.1. *For $G \in \boldsymbol{W}$, with weak unit u_G, there are a unique compact Hausdorff space YG and a lattice embedding $\eta_G : G \hookrightarrow D(YG)$, such that $\eta_G(u_G) = \mathbf{1}$ (the constant function equal to 1 on YG), with $\eta_G(G)$ separating the points of YG, closed under the pointwise operations of $D(YG)$*

requisite to making $\eta_G(G)$ an ℓ-group; G and $\eta_G(G)$ are isomorphic $\boldsymbol{W}$-objects. Moreover, if $\phi : G \longrightarrow H$ is a $\boldsymbol{W}$-homomorphism, that is, a lattice homomorphism for which $\phi(u_G) = u_H$, then there exists a unique continuous map $\tau : YH \longrightarrow YG$, such that for every $g \in G$, $\eta_H(\phi(g)) = \eta_G(g) \cdot \tau$. We suppress notation and write $G \leq D(YG)$ and $\phi(g) = g \cdot \tau$.

A partial view of **hoW** appears as a Hasse diagram in [11] pg. 167, in which are depicted several distinct chains, which are faithfully indexed by the regular cardinals. While **hoW** is huge (a proper class or larger, depending on one's definition of "proper class") for the purpose of providing distinction of present interests, we require, perhaps, only four examples, which we now describe, in terms of the Yosida Representation $G \leq D(YG)$.

Example 1.2. (1) The local reflection *loc*: $f \in D(YG)$ is "locally" in G if there exists a finite subcover $\mathscr{U}$ of YG and a collection of elements of G, $\{g_U | U \in \mathscr{U}\}$ such that $f|_U = g_U$ for every $U \in \mathscr{U}$. The local hull of G, $locG$, is the ℓ-subgroup in $D(YG)$ comprised of all the elements of $D(YG)$ which are locally in G. The hull operator *loc* is a reflection in **W**, and $YlocG = YG$. For additional, information see [24].

(2) The maximum essential reflection c^3: Define $G^{-1}(\mathbb{R}) \equiv \{g^{-1}(\mathbb{R}) \,|\, g \in G\}$. $G^{-1}(\mathbb{R})$ is a filter base of dense open subsets of YG and $G^{-1}(\mathbb{R})_\delta$, is the filter base of all countable intersections of elements of $G^{-1}(\mathbb{R})$ (which are dense by the Baire Category Theorem). For any filter base of dense sets $\mathscr{F}$ on a space X, define $C[\mathscr{F}] \equiv \varinjlim\{C(F) \,|\, F \in \mathscr{F}\}$. $C[\mathscr{F}]$, which is the direct limit in **W**, is $\bigcup_{F \in \mathscr{F}} C(F)/\sim$, where $f_1 \sim f_2$ if and only if f_1 and f_2 agree on the intersection of their domains. Then,

$$G \leq C[G^{-1}(\mathbb{R})] \leq C[G^{-1}(\mathbb{R})_\delta] \equiv c^3 G.$$

In [3]) the authors demonstrate that c^3G is the maximum essential reflection in **W**. There is more information in many other places such as [5] and its references. For future reference, we note the following example. If G_0 is the eventually polynomial functions on $\mathbb{N}$, then $YG_0 = \alpha\mathbb{N}$, the one point compactification of $\mathbb{N}$, $c^3G_0 = C(\mathbb{N})$ and $Yc^3G_0 = \beta\mathbb{N}$, the Čech-Stone compactification of $\mathbb{N}$.

(3) The essential completion e: Any compact Hausdorff space X has its "absolute" (Gleason cover, projective cover) which is an irreducible surjection $\pi : aX \longrightarrow X$, where aX is compact extremally disconnected. For

$G \in \mathbf{W}$ one has $YG \xleftarrow{\pi} aYG$ and if one defines $eG \equiv D(aYG)$ and $\phi : G \longrightarrow eG$ by $\phi(g) = \pi \cdot g$, then $G \overset{e}{\leq} eG$. There are details to this, of course, some of which is discussed in [9]. This is all a version of Conrad's description of the essential completeness and completion in [12]. The following observation is useful: $D(aX) \approx C[\mathscr{G}(X)_\delta]$, where $\mathscr{G}(X)_\delta$ is the filter base of dense G_δ's in X [16].

(4) The Dedekind completion of the divisible hull c: This is

$$cG \equiv \{f \in eG \,|\, |f| \leq g \text{ for some } g \in G\},$$

which is the ℓ-group ideal in eG that is generated by G (in [14], 54.23 and 57.16). Since $BcG = BeG$, where B is bounded monocoreflection in $\mathbf{W}$ (see below), $YcG = YeG = aYG$.

Here is a preliminary result about the hull operators in Example 1.2.

Proposition 1.3. *The hull operators in Example* 1.2 *are related as follows.*
(a) $loc < c^3 < e$.
(b) $loc < c < e$.
(c) *c and c^3 are incomparable.*

Proof. Observe that $\leq$ holds in (a) and (b). [[24]] says that any $\mathbf{W}$-object G, which is the "$\mathbf{W}$-part" of an f-ring with identity has $locG = G$. As any c^3-object is such a G, $locG \leq c^3G$. As to the $loc \leq c$ case, $G = cG$ implies that G is projectable ([14]) and projectable implies local ([22]). Recall, if $h_1 \leq h_2$ to prove $h_1 < h_2$, it suffices to show that there exists an $G \in \mathbf{W}$ such that $h_1G < h_2G$.
(a) Since any $\mathbf{W}$-object G which is the "$\mathbf{W}$-part" of an f-ring with identity has $G = locG$, G_0 from (2) in Example 1.2 satisfies $G_0 = locG_0 < c^3G_0$. As to the other inequality in (a), if X is compact, then $YC(X) = X$ and $C(X) = c^3C(X)$. If X is also infinite, then $D(aX)$ contains unbounded functions (as a compact infinite extremally disconnected space is not a P-space [18]). Consequently, $C(X) = c^3C(X) < D(aX) = eC(X)$.
(b) $loc < c$. Since for every G, $YlocG = YG$, whereas, $YcG = aYG$, one may proceed as in (a) but with X compact, infinite and not extremally disconnected. On the other hand, since for every compact X, $cC(X) = BeC(X) = C(aX)$ and, usually, $C(aX) < D(aX)$, it follows that $c < e$.
(c) We demonstrate that there exists G_1 and G_2 such that $c^3G_1 < cG_1$

and $cG_2 < c^3G_2$. If $G_1 = C(X)$, for X compact, not extremally disconnected, then $G_1 = c^3G_1 < C(aYG_1) = cG_1$. Now take Y to be an infinite, extremally disconnected and pick $g \in D(Y)$ unbounded and $g(y) > 0$ for every $y \in Y$. Let G_2 be the sub-ℓ-group of $D(Y)$ generated by $C(Y)$ and g. Then, $cG_2 = \{f \in D(Y) \,|\, |f| \leq ng \text{ for some } n \in \mathbb{N} \text{ and } g \in G_2\}$ and $c^3G_2 = C(g^{-1}\mathbb{R})$. Consequently, $cG_2 < c^3G_2$. □

We now review the bounded monocoreflection in $\mathbf{W}$. For $G \in \mathbf{W}$, with weak unit u_G, define

$$BG = \{g \in G \,|\, |g| \leq nu_G \text{ for some } n \in \mathbb{N}\}.$$

$BG = G \cap C(YG)$ and u_G is a strong unit for the $\mathbf{W}$-object BG. Define $B\mathbf{W} = \{G \in \mathbf{W} \,|\, BG = G\}$. Then $B\mathbf{W}$ is a monocoreflective subcategory of $\mathbf{W}$, where the functor $B : \mathbf{W} \longrightarrow B\mathbf{W}$ is the monocoreflector. [11] investigates the interactions of B with hull operators with a quite different thrust than the present paper. Section 3 there lists various properties of B from which one, easily, infers the following.

Proposition 1.4. *Suppose $h \in \mathbf{ho}\,\mathbf{W}$. For any $G \in \mathbf{W}$, one has the following commutative diagram in which each arrow is an essential embedding:*

$$\begin{tikzcd}
hBG & & & \\
& & BhG \arrow[dr] & \\
BG \arrow[uu] \arrow[urr] \arrow[r] & G \arrow[rr] & & hG
\end{tikzcd}$$

Here Bh (respectively, hB) is the composition of the functions h and then B (respectively, B and then h).

Throughout we make use of Proposition 1.4 without mention.

The subject of **hoC** for various categories **C**, has a large literature. See the bibliographies of our previous papers [9], [10], and [11]. We won't totally replicate those bibliographies here but do mention Conrad's seminal papers [12], [13] and Martinez's survey [27]. There are various "nearly" dual situations, including **coComp**, covering operators for compact spaces (the largest being the "a" mentioned in (3) of Example 1.2). The connection between **hoW** and **coComp** are studied in [9] and [28]. By itself **coComp** is examined in [21] and [30], [29]. We apologize to authors not mentioned.

2 Words

Definition 2.1. (a) "x" is a variable ranging in $\mathbf{hoW}$, and is a function from $\mathbf{hoW} \longrightarrow \mathbf{W}^{\mathbf{W}}$, whose action at $x = h$ is the function $h \in \mathbf{W}^{\mathbf{W}}$ (whose action at $G \in \mathbf{W}$ is $G \mapsto hG$).

(b) The bounded monocoreflection $B \in \mathbf{W}^{\mathbf{W}}$, and may be construed to be the constant function $\mathbf{hoW} \longrightarrow \mathbf{W}^{\mathbf{W}}$, whose action is $h \mapsto B$ for every h. Then, the composition of functions $w(x) = xB$, x after B, is a function from $\mathbf{hoW} \longrightarrow \mathbf{W}^{\mathbf{W}}$, whose action at $x = h$ is $G \mapsto hBG$. A (general) word $w(x)$ or w is a function, which is a finite string of successive compositions, $w(x) = y_n \cdot y_{n-1} \cdots y_2 \cdot y_1$, where $n \in \mathbb{N}$, for $1 \leq i \leq n$, $y_i = h$ or B and y_{i+1} is after y_i for $1 \leq i < n$. Throughout, "Word" is the set of words, that is, such functions.

(c) Let $w_1, w_2 \in$ Word. The ordering $w_1 \leq w_2$ is "pointwise" : for every $h = x$, $w_1(h) \leq w_2(h)$; that is, for every $G \in \mathbf{W}$, $w_1(h)G \leq w_2(h)G$, where the last $\leq$ signifies "is embedded as a $\mathbf{W}$-subobject of". The multiplication w_2w_1 is composition of functions, w_2 after w_1, which is expressible as concatenation of the strings. Note, the example $w(x) = xB$ is the product w_2w_1, where $w_2 = x$, $w_1 = B$.

Theorem 2.2. *There are exactly* 6 *words, listed and ordered* (*per* 2.1) *as in the following picture in which the order is* $\uparrow$

$$\begin{tikzcd} & x \arrow[d, no head] & \\ & xBx \arrow[dl, no head] \arrow[dr, no head] & \\ Bx \arrow[dr, no head] & & xB \arrow[dl, no head] \\ & BxB \arrow[d, no head] & \\ & B & \end{tikzcd} \qquad (\star)$$

Consequently, Word is order isomorphic to $F(2)$.

Before proving Theorem 2.2, we note that Word has more algebraic structure as described below. This is intriguing and could well inform some of the issues articulated in section 4. But, we don't know about that, so we shall omit the proof of Corollary 2.3 below (which we have only achieved through tedious and lengthy, case-by-case verifications).

A *partially ordered semigroup* is an algebraic system $(S, \cdot, \leq)$ for which $(S, \cdot)$ is a semigroup and $(S, \leq)$ is a partially ordered set, and $a \leq b$ implies that

$$ca \leq cb$$

$$ac \leq bc$$

for all $c \in S$, and is an *ℓ-semigroup* if $(S, \leq)$ is a lattice and

$$(\vee) \quad c(a \vee b) = (ca \vee cb), \; \forall\, c,$$

and is an *$\ell\ell$-semigroup* if also

$$(\wedge) \quad c(a \wedge b) = (ca \wedge cb), \forall\, c.$$

See [17], where it is shown that $(\vee)$ does not imply $(\wedge)$. We have coined the term "$\ell\ell$".

Corollary 2.3. *Word with the multiplication as defined in Definition* 2.1 *is an $\ell\ell$-semigroup. It is very non-commutative and without an identity on the left or the right.*

We now prove Theorem 2.2.

Proof. The partial order and multiplication throughout this proof are as in 2.1. We shall use the following features of B and any $h \in \mathbf{hoW}$.

(i) B is decreasing, preserves $\leq$, and $BG \overset{\text{e}}{\leq} G$ for $G \in \mathbf{W}$.

(ii) h is increasing, preserves $\overset{\text{e}}{\leq}$, and $G \overset{\text{e}}{\leq} hG$ for $G \in \mathbf{W}$.

It follows from the previous items, that the partial order in Word is as depicted in $(\star)$. We use this order to prove the rest of the theorem by establishing the following.
(1) Each of the 6 words in $(\star)$ is idempotent; that is, $w \cdot w = w$ or $w^2 = w$.
(2) Word $\subseteq$ the set of the six words in $(\star)$.
(3) Each word in $(\star)$ is distinct.

Note, any statement $w(x) = \cdots$, means $w(h) = \cdots$ for all h. When convenient we replace the x with h in the following:
(1) is a result of the following arithmetic verifications.

(i) $B^2 = B$ and $h^2 = h$. This is obvious.

(ii) $(Bh)^2 = Bh$. If $G \in \mathbf{W}$, then $BhG \overset{e}{\leq} hG$, and hence, $hBhG \overset{e}{\leq} h^2G = hG$. Applying B to second inequality yields $BhBhG \leq BhG$. On the other hand, since $BhG \leq hBhG$, applying B to the previous inequality yields the result.

(iii) $(hB)^2 = hB$. If $G \in \mathbf{W}$, then $BhBG \overset{e}{\leq} hBG$, and hence, $hBhBG \overset{e}{\leq} h^2BG = hBG$. On the other hand, $hBG = hB(BG) \leq hB(hBG) = hBhBG$.

(iv) $(hBh)^2 = hBh$. By the idempotent feature of h and Bh,

$$(hBh)(hBh) = (hBhBh) = hBh.$$

(v) $(BhB)^2 = BhB$. By the idempotent feature of B and hB,

$$(BhB)(BhB) = (BhBhB) = BhB.$$

(2) Consider a general word $w(x) \neq B$ or x. Then, $w(x) = y_n \cdot y_{n-1} \cdots y_2 \cdot y_1$, with at least two different terms (that is, both B and x appear), with B's and x's alternating. Replacing x by h and utilizing (1) yields that there exists a $1 \leq p \in \mathbb{N}$ such that (by (ii) of (1)).

$$w(h) = \begin{cases} (hB)^p B = hBB = hB, or \\ (Bh)^p h = Bhh = Bh, or \\ (hB)^p h = hBh, or \\ (Bh)^p B = BhB. \end{cases}$$

(3) We want to show

$$\begin{aligned} &(a)\, B < BxB < Bx < xBx < x \\ &(b)\, BxB < xB < xBx \\ &(c)\, Bx \neq xB. \end{aligned}$$

Recall $w_1(x) < w_2(x)$ means that $w_1(x) \leq w_2(x)$ and there exists an $h \in$ **hoW** for which $w_1(h) < w_2(h)$; that is, there exists a $G \in \mathbf{W}$ such that

$w_1(h)G < w_2(h)G$. For the various assertions above only two hull classes are required (probably as F(2) is generated by two elements). Any two of disparate character will suffice (as commented parenthetically below with reference to terminology of section 3). We choose c^3 and e, respectively, the maximum essential reflection and maximum hull operator. *Per* our observation above, we calculate for every w, $w(c^3)$ and $w(e)$ and indicate appropriate G's in $\mathbf{W}$; recall from Example 1.2, $c^3G = C[G^{-1}(\mathbb{R})_\delta]$ and $eG = D(aYG)$.

w	$w(c^3)G$	$w(e)G$
B	$BG \leq C(YG)$	$BG \leq C(YG)$
BxB	$Bc^3BG = C(YG)$	$BeBG = C(aYG)$
Bx	$Bc^3G = C^*[G^{-1}(\mathbb{R})_\delta]$	$BeG = C(aYG)$
xB	$c^3BG = C(YG)$	$eBG = D(aYG)$
xBx	$c^3Bc^3G = C^*[G^{-1}(\mathbb{R})_\delta]$	$eBeG = D(aYG)$
x	$c^3G = C[G^{-1}(\mathbb{R})_\delta]$	$eG = D(aYG)$

It follows from the above calculations along with the appropriate choices for G that:
$B < BxB$, for $x = c^3$ or $x = e$ (in fact, $B = BxB$ is known only for $x = Id$ or loc).
$BxB < Bx$, for $x = c^3$ (or any $x = h$, which is PB^{op}).
$BxB < xB$, for $x = e$ (or any $x = h$, which is not PB).
$Bx < xBx$ for $x = e$ (or any $x = h$, which is not PB).
$xB < xBx$ for $x = c^3$ (or any $x = h$, which is not PB^{op}).
$xBx < x$ for $x = c^3$ (or any $x = h$, which is PB).
To see $xB \neq Bx$, observe if $h = c^3$ (or any h PB but not PB^{op}), then $hB < Bh$. On the other hand, if $h = e$ (or any h *antiPB*), then $Bh < hB$. $\square$

Remark 2.4. In connection with the last item of the proof above ($xB \neq Bx$), we don't know if for every h, either $Bh \leq hB$ or $Bh \geq hB$. See Sections 5 and 6 below for other versions of the same question.

3 Equations in Word

An *equation* is an expression $E : w_1 = w_2$, where $w_i \in$ Word. $x = h$ *satisfies* E if $w_1(h) = w_2(h)$ (which means for every $G \in \mathbf{W}$, $w_1(h)G = w_2(h)G$), and is denoted by $h \models E$. Since there are 6 words, there are $6 \times 6 = 36$ possible equations. However, since $w_1 = w_2$ is the same as $w_2 = w_1$, there are only 18 to consider. Moreover, since any h is increasing, it follows that no h satisfies $x = B \cdots$. Consequently, this eliminates $x = B$, $x = BxB$ and $x = Bx$ and leaves only 15 for consideration. Throughout the phrase "E is an equation" refers to one of these 15.

Definition 3.1. Let $h \in \mathbf{hoW}$.

(a) h *preserves boundedness* or is PB if $hB = BhB$. As demonstrated in Theorem 3.4 in [9], h is PB if and only if $Bh = hBh$ if and only if $h[B\mathbf{W}] \subseteq B\mathbf{W}$.

(b) h is *opposite of* PB or is PB^{op} if $hB = hBh$. This holds if and only if $Bh = BhB$.

(c) h *commutes with* B or is CB if $Bh = hB$. This holds if and only if $BhB = hBh$ if and only if h is PB and PB^{op}.

(d) h is $antiPB$ if $h = hB$.

(e) h is *almost* B or alB if $B = hBh$.

(f) h is M^- (respectively, M^+) if $B = BhB$ (respectively, $h = hBh$), where the "M" stands for mysterious.

The symbols PB etc. will also be used for the class $\{h \mid h \text{ is } PB\}$ etc. In addition, we note, $CB = PB \bigcap PB^{op}$, $PB \bigcap M^+ = \emptyset$, $PB^{op} \bigcap M^+ = antiPB$.

The facts inserted into Definitions 3.1 are easily shown using repeatedly the identity $w^2 = w$ for every w (item (1) in the proof of Theorem 2.2). Some of the names and reasons for existence, etc. appear in our various papers in the references, in particular [11].

We now present the chart of equations, with explanation to follow. Here N signifies the equation is never satisfied (e.g., $B = x$), T signifies a tautology that is, $w = w$. Below the diagonal is the reflection of above and the diagonal is $w_1 = w_2$ "equals" $w_2 = w_1$. The arrows signify implications (that is, inclusion of class) and we do not know if any or all arrows in the B-row reverse. The only B-row examples are Id and loc, which are alB. The ? in the B-row remain un-named and we do not know if $antiPB$ implies M^+. All other arrows do not reverse. We note that there is the set of "basic" equations $\mathscr{B} = \{M^-, PB, PB^{op}, M^+\}$: each other equation is a conjunction of these. This and the implications can be easily shown (some previously asserted in 3.1).

Table 1: Equations

$w_1 \backslash w_2$	B	BxB	Bx	xB	xBx	x
B	T	M^-	$?$	$?$	alB	N
BxB		T	PB^{op}	PB	CB	N
Bx			T	CB	PB	N
xB				T	PB^{op}	$antiPB$
xBx					T	M^+
x						T

Proposition 3.2. *Recall that E refers to the* 15 *viable equations.*

(1) *For every equation E, there are $p, n \in$ **ho*W*** for which $p \models E$, $n \not\models E$.*

(2) *There does not exist an h such that $h \models E$ for every E.*

Proof. (1) Since by Theorem 2.2 the words are all distinct, it follows that for every E there exists an $n \in \mathbf{hoW}$ such that $n \not\models E$. $p = Id$ or loc is alB, thus p satisfies E for all $E \neq antiPB$ or M^+. $p = e$, the essential completion, satisfies $E = antiPB$ and M^+.

(2) $M^+ \bigcap PB = \emptyset$. $\square$

4 Quotients; a partition of hoW

Especially in this section a picture is worth many words. Here is Word with the vertices relabelled to reduce notation.

Word* 4.1*:

$$\begin{tikzcd}[column sep=small]
 & 1 = x \arrow[d, dash] & \\
 & \chi = xBx \arrow[dl, dash] \arrow[dr, dash] & \\
Bx = \lambda \arrow[dr, dash] & & \rho = xB \arrow[dl, dash] \\
 & \alpha = BxB \arrow[d, dash] & \\
 & 0 = B &
\end{tikzcd}$$

Definition 4.2. Fix $h \in \mathbf{hoW}$ a surjection Word $\xrightarrow{\sigma_h} Q(h)$ is defined by the equations E that h satisfies: $\sigma_h(w_1) = \sigma_h(w_2)$ means $w_1(h) = w_2(h)$. We define $\sigma_h(w_1) \leq \sigma_h(w_2)$ exactly when $w_1 \leq w_2$.

Proposition 4.3. *Fix $h \in \mathbf{ho}\,\boldsymbol{W}$. $Q(h)$ is a lattice and σ_h is a lattice homomorphism. σ_h obeys the following laws:*

$$\begin{aligned}
\langle PB\rangle &\ \ \sigma_h(\alpha) = \sigma_h(\rho) \text{ if and only if } \sigma_h(\lambda) = \sigma_h(\chi)\\
\langle PB^{op}\rangle &\ \ \sigma_h(\alpha) = \sigma_h(\lambda) \text{ if and only if } \sigma_h(\rho) = \sigma_h(\chi)\\
\langle h \neq B\cdots\rangle &\ \ \sigma_h(\lambda) \neq \sigma_h(1).
\end{aligned}$$

Proof. By definition, $Q(h)$ is partially ordered and σ_h preserves order. The facts that $Q(h)$ is a lattice and σ_h a lattice homomorphism reduce to the issues that the following two equations

$$(E1)\ Bh \vee hB = hBh$$

$$(E2)\ Bh \wedge hB = BhB$$

hold in $Q(h)$. It is easy to see that if $Bh \neq hB$ in $Q(h)$, then $E1$, and $E2$ hold. In the case, $Bh = hB$ in $Q(h)$, then $Q(h)$ is the 3-chain (see Figure 6 below), and again we have equality. $\square$

Remark 4.4. (a) The notation $\langle PB\rangle$ for the law is to avoid confusion with property PB that a hull operator h might have (the class $PB = \{h \mid h \text{ is } PB\}$).

(b) We don't know if any onto lattice homomorphism Word $\xrightarrow{\sigma} Q$, which satisfies the laws, is in fact, a σ_h for some $h \in \mathbf{hoW}$. In particular, we don't know if $\sigma = Id$ is a σ_h; such an h would satisfy no equation and in particular, $h \notin PB \bigcup PB^{op}$. We come back to this in 4.9, 5.2 and 5.3.

Behind the σ_h are the quotients by equations (and combinations of equations). We draw a picture of the equation $E : w_1 = w_2$ and the resultant quotient Word $\xrightarrow{\sigma(E)} Q(E)$ by encircling together the vertices w_1, w_2 in 4.1. For the "basic equations" $\mathscr{B} = \{M^-, PB, PB^{op}, M^+\}$: The proof for the following proposition is straightforward and omitted.

Proposition 4.5. *If $h \in \mathbf{ho}\,\boldsymbol{W}$, then, either h satisfies no equation, hence $\sigma_h = Id$ (see Remark 4.4 (b)), or h satisfies some equation E in $\mathscr{B}$ and therefore $\sigma(E)$ is a first factor of σ_h (that is, $Q(h)$ is a quotient of $Q(E)$).*

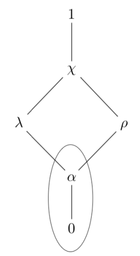

Figure 1: $\sigma(M^-)$

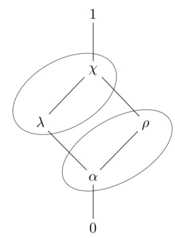

Figure 2: $\sigma(PB)$

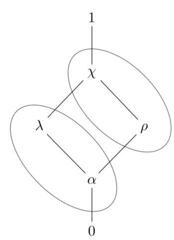

Figure 3: $\sigma(PB^{op})$

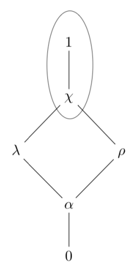

Figure 4: $\sigma(M^+)$

We refine the process further towards our partition of **hoW** with notation and pictures, which we trust are obvious by now.

Theorem 4.6. *Referring to Figures 5-8.*
(a) *The four classes are disjoint.*
(b) *Each picture is realized from* $\mathbf{ho}\,\boldsymbol{W}$ (*that is, there is a* (*or many*) σ_h)*; each of the four classes is nonempty.*

Proof. (a) The only perhaps non-obvious disjointness is $(CB\backslash PB)\cap antiPB =$

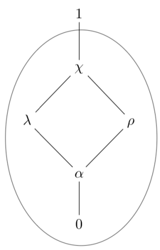

Figure 5: $\sigma(alB)$

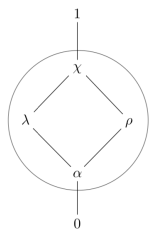

Figure 6: $\sigma(CB \setminus alB)$

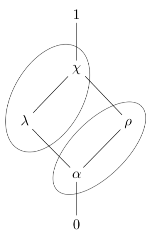

Figure 7: $\sigma(PB \setminus CB)$

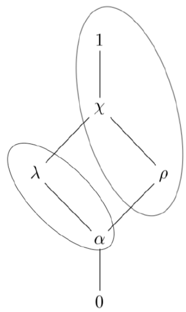

Figure 8: $\sigma(antiPB)$

$\emptyset$; in fact, $PB \cap antiPB = \emptyset$.

(b) (i) Id and $loc \in alB$ (and are the only ones we know).

(ii) c, the Dedekind completion of the divisible hull, has $c \in CB \setminus alB$.

(iii) $c^3 \in PB \setminus CB$.

(iv) $e \in antiPB$.

□

We do not know if every σ_h is one of the above four. Those four are all chains. We don't know if every $Q(h)$ is a chain (see Sections 5 and 6).

We have the Venn Diagram:

Here, **hoW** is the disc.

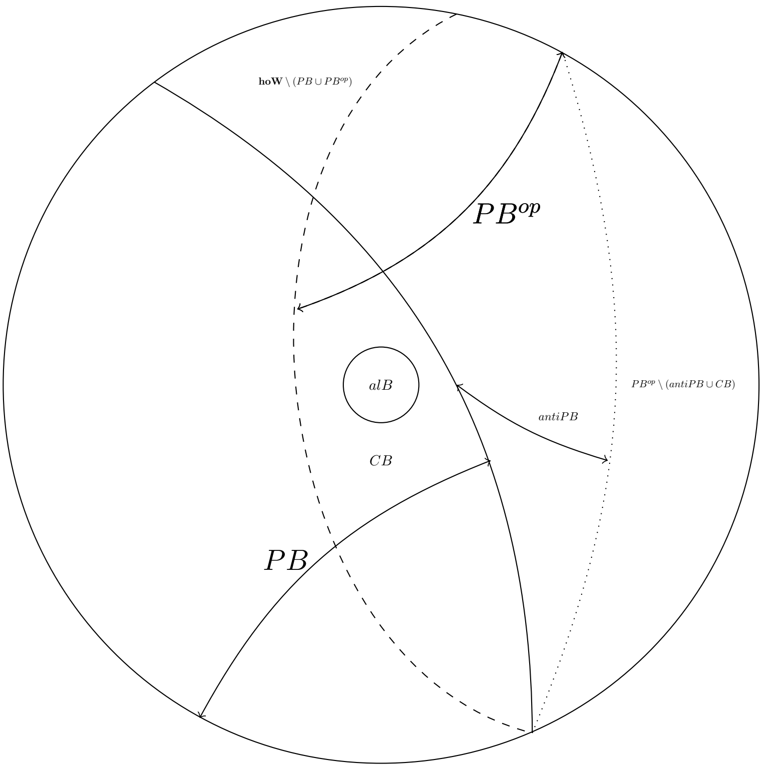

Figure 9:

PB is to the left of the concave-left solid line.

PB^{op} is to the right of the concave-right dashed line.

$CB = PB \cap PB^{op}$.

$alB \subseteq CB$, and $antiPB \subseteq PB^{op}$ and is to the left of the concave-left dotted line.

Remark 4.7. In the Venn diagram, we see six disjoint regions comprising **hoW**. Our knowledge of the size of the regions is

(i) $\mathbf{hoW} \setminus (PB \cup PB^{op})$ is not known to be nonempty. See sections 5, 6 below.

(ii) alB contains at least two elements, Id and loc. We know no more. See section 6 below.

(iii) $PB \setminus CB$ contains many, not all, essential reflections, and is at least infinite (revealed from the parameterizations of these in [20]). We are sure more can be said but depart the topic for now.

(iv) Each of the following contains chains faithfully indexed by the regular cardinals. $CB \setminus alB$ and $antiPB$ (see section 2 of [11]); $PB^{op} \setminus (CB \cup antiPB)$ (see [8]).

Some further subdivision of the regions can be found in [9] (especially of PB), and [10] and [11] (especially of CB).

5 PB and PB^{op}

Earlier, we raised three questions (among others):
(1) In 2.4. For every hull operator h does $Bh \leq hB$ or $Bh \geq hB$?
(2) After 4.6. Is every $Q(h)$ a chain?
(3) In 4.4 and after 4.6. Does $\mathbf{hoW} = PB \bigcup PB^{op}$?

The following says that these are the same question, even "locally at h." We do not know the answers, and shall pick apart the proof in the hopes of informing the situation.

Theorem 5.1. *For $h \in \mathbf{ho}\,\boldsymbol{W}$, with its Word $\xrightarrow{\sigma_h} Q(h)$, these are equivalent.*

(1) *$\sigma_h(\lambda)$ and $\sigma_h(\rho)$ are comparable (that is, $Bh \leq hB$ or $hB \leq Bh$).*

(2) *$h \in PB \cup PB^{op}$.*

(3) *$Q(h)$ is a chain.*

The following triviality shows Theorem 5.1 (1) $\iff$ (3).

Lemma 5.2. *Suppose Q is partially ordered and Word $\xrightarrow{f} Q$ is a surjection which preserves order. $f(\lambda)$ and $f(\rho)$ are comparable if and only if Q is a chain.*

Proof. Since f preserves order, we have:

$$f(0) \leq f(\alpha) \leq A \leq f(\chi) \leq f(1),$$

where A denotes $f(\lambda)$ or $f(\rho)$. The assertion is obvious. $\square$

Lemma 5.3. *Suppose Q is partially ordered and Word $\xrightarrow{f} Q$ is a surjection which preserves order and satisfies the laws $\langle PB\rangle$ and $\langle PB^{op}\rangle$. The following are equivalent.*

(1) *Q is a chain.*
(2) *There exists $w \neq \lambda$ with $f(w) = f(\lambda)$.*
(3) *There exists $w \neq \rho$ with $f(w) = f(\rho)$.*
(4) *On Word$\setminus \{0, 1\}$, f is not one-to-one.*

Proof. (1) $\Longrightarrow$ (4): Obviously, if (4) fails, then (1) fails.
(4) $\Longrightarrow$ (2): If (2) fails but (4) holds, then the truth of (4) must be exemplified by one of: (a) $f(\alpha) = f(\chi)$; (b) $f(\rho) = f(\chi)$; (c) $f(\alpha) = f(\rho)$. But if (a) holds, then (2) holds, since f preserves order. If (b) holds, then $f(\alpha) = f(\lambda)$ by $\langle PB^{op}\rangle$. If (c) holds, then $f(\chi) = f(\lambda)$ by $\langle PB\rangle$. Thus, (4) fails.
(4) $\Longrightarrow$ (3): This is symmetric to (4) $\Longrightarrow$ (2).
(2) $\Longrightarrow$ (1): We show that (2) implies $f(\lambda)$ and $f(\rho)$ are comparable (then apply 5.1). Suppose $f(w) = f(\lambda)$ for some $w \neq \lambda$. Since f preserves order w can be α, ρ, or χ. If $f(\rho) = f(\lambda)$, we are done. If $f(\alpha) = f(\lambda)$, then $f(\lambda) \geq f(\rho)$ since $f(\alpha) \geq f(\rho)$ (f preserves order). If $f(\chi) = f(\lambda)$, then $f(\rho) \geq f(\lambda)$ since $f(\rho) \geq f(\chi)$ (f preserves order).
(3) $\Longrightarrow$ (1): This is symmetric to (2) $\Longrightarrow$ (1). $\square$

The following proves (more than) Theorem 5.1 (1) $\Longleftrightarrow$ (2) and completes the proof of 5.1.

Theorem 5.4. *Suppose $h \in \mathbf{ho}\,\boldsymbol{W}$, with its $\sigma \equiv \sigma_h$: Word $\twoheadrightarrow Q(h)$.*

(1) *$\sigma(\lambda) \leq \sigma(\rho)$ (that is, $Bh \leq hB$) if and only if $h \in PB^{op}$.*
(2) *$\sigma(\lambda) \geq \sigma(\rho)$ (that is, $Bh \geq hB$) if and only if $h \in PB$.*

Proof. The implications $\Longleftarrow$ are clear.
(1) $\Longrightarrow$ Suppose $\sigma(\lambda) \leq \sigma(\rho)$. By Lemma 5.2 $Q(h)$ is a chain, so by Lemma 5.3, there exists $w \neq \lambda$ with $\sigma(w) = \sigma(\lambda)$. Since σ preserves order we can have $w = \alpha$, ρ or χ. $\sigma(\alpha) = \sigma(\lambda)$ is $BhB = Bh$: h is PB^{op}. $\sigma(\chi) = \sigma(\lambda)$ implies $\sigma(\lambda) = \sigma(\rho)$ (since σ preserves order), and $\sigma(\rho) = \sigma(\chi)$ is $h \in CB \subseteq PB^{op}$.
(2) $\Longrightarrow$ Is symmetric. $\square$

We state the obvious in Corollaries 5.5 and 5.6.

Corollary 5.5. *The following are equivalent.*

(a) *For every $h \in \mathbf{ho}\,\boldsymbol{W}$, either $Bh \leq hB$ or $Bh \geq hB$.*

(b) $\mathbf{ho}\,\boldsymbol{W} = PB \cup PB^{op}$.

(c) *For every $h \in \mathbf{ho}\,\boldsymbol{W}$, $Q(h)$ is a chain.*

We don't know if the conditions in Corollary 5.5 hold. Here is "if not."

Corollary 5.6. *For $h \in \mathbf{ho}\,\boldsymbol{W}$, $Q(h)$ is not a chain if and only if σ_h is exactly one of*

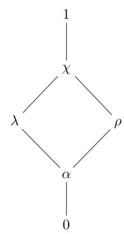

Figure 10: No Equation

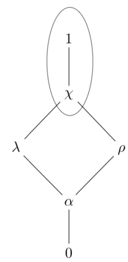

Figure 11: M^- only

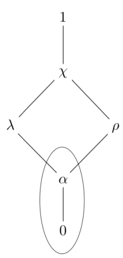

Figure 12: M^+ only

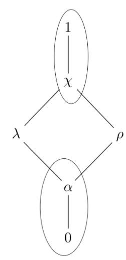

Figure 13: M^+ & M^-

Proof. These are the choices for which (4) in Lemma 5.3 fails. $\square$

6 Remarks and Questions

(1) For every h, $2 \leq |Q(h)| \leq 6$. The first $\leq$ is from the law $\langle h \neq B \cdots \rangle$ (the same as $\sigma_h(\lambda) \neq \sigma_h(1)$).

(2) It follows from Theorem 5.1 that if $Q(h)$ is a chain, then $|Q(h)| \leq 4$. Instantiation of chains with $|Q(h)| = 2, 3, 4$ appears in 4.6 (b). Some further thought shows certain $4-$, $3-$, and $2-$ chains are not $\sigma(h)$s (and some we do not know). We omit the details. The (apparent) possibilities for (non-chains) $|Q(h)| = 5, 6$, with their equations, are depicted in Corollary 5.6.

Here are further mysteries. It seems likely that every question is related to every other question.

(3) The B-row in Table 1. We only know Id, $loc \in alB$. We know essentially nothing about the equation M^- (cf. 5.6). A related question: does there exists an $h \in \mathbf{hoW}$ for which $Id < h < loc$?

(4) The equation M^+. It is clear that (in terms of classes) $M^+ \cap PB = \emptyset$ (from the law $\langle h \neq B \cdots \rangle$) and $antiPB \subseteq M^+$. Is "$\supseteq$" valid? (cf. 4.7 and 5.3).

(5) Does $PB =$ reflective$\cup CB$? Our preliminary sorting through various constructions of PB's in [9], [10], and [11] has failed to answer this.

(6) The question does $\mathbf{hoW} = PB \cup antiPB$ was raised in [9] (sections 3, 7). This is answered negatively in [8].

(7) Our present central question is does $\mathbf{hoW} = PB \cup PB^{op}$, raised here in 4.7 (i) and amplified in section 5. (This is a refinement of the question in (6) above.)

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References

[1] Anderson, M.E. and Feil, T.H., "Lattice-ordered Groups: an Introduction", Springer Science & Business Media, 2012.

[2] Balbes, R. and Dwinger, P., "Distributive lattices", New York, 1974.

[3] Ball, R. and Hager, A.W., *Algebraic extensions of an archimedean lattice-ordered group* I, J. Pure Appl. Algebra 85(1) (1993), 1-20.

[4] Ball, R., Hager, A.W., and Neville, C., *The Quasi-FK cover of Compact Hausdorff Space and the AC-Ideal Completion of an Archimedean l-Group*, General Top. Appl.: Proceedings of the 1988 Northeast Conference 123, 1990.

[5] Banaschewski, B. and Hager, A., *Representation of H-closed monoreflections in archimedean l-groups with weak unit*, Categ. General Alg. Struct. Appl. 9(1) (2018), 1-13.

[6] Bigard, A., Keimel, K., and Wolfenstein, S., "Groupes et Anneaux Réticulés", Springer, 2006.

[7] Birkhoff, G., "Lattice theory", American Mathematical Society, 1940.

[8] Carrera, R., *Various Completeness of an Archimedean Lattice-ordered Group*, 2014.

[9] Carrera, R. and Hager, A.W., *On hull classes of l-groups and covering classes of spaces*, Math. Slovaca 61(3) (2011), 411-428.

[10] Carrera, R.E. and Hager, A.W., *B-saturated hull classes in ℓ-groups and covering classes of spaces*, Appl. Categ. Structures, 23(5) (2015), 709-723.

[11] Carrera, R.E. and Hager, A.W., *Bounded equivalence of hull classes in archimedean lattice-ordered groups with unit*, Appl. Categ. Structures 24(2) (2016), 163-179.

[12] Conrad, P., *The essential closure of an archimedean lattice-ordered group*, Duke Math. J. 38(1) (1971), 151-160.

[13] Conrad, P., *The hulls of representable l-groups and f-rings*, J. Austral. Math. Soc. 16(4) (1973), 385-415.

[14] Darnel M. R., "The Theory of Lattice-Ordered Groups", Pure Appl. Math. 187, Marcel Dekker, 1995.

[15] Engelking, R."Outline of General Topology", North-Holland Pub. Co., 1968.

[16] Fine, N., Gillman, L., and Lambek, J., *Rings of Quotients of Rings of Functions*, McGill University Press, 1996.

[17] Fuchs, L., "Partially Ordered Algebraic Systems", Courier Corporation, 2011.

[18] Gillman, L. and Jerison, M., "Rings of Continuous Functions", Courier Dover Publications, 2017.

[19] Gleason, A., *Projective topological spaces*, Illinois J. Math. 2(4A) (1958), 482-489.

[20] Hager, A.W., *Algebraic closures of l-groups of continuous functions*, in “Rings of Continuous Functions”, C. Aull, Ed., Lecture Notes Pure Appl. Math. 95 (1985), Marcel Dekker, 165-194.

[21] Hager, A.W., *Minimal covers of topological spaces*, Ann. New York Acad. Sci. 552(1) (1989), 44-59.

[22] Hager, A.W. and Martinez, J.,“α-projectable and laterally α-complete archimedean lattice-ordered groups”, in S. Bernahu (ed.), Proc. Conf. on Mem. of T. Retta, Temple U., PA/Addis Ababa (1995), Ethiopian J. Sci., 73–84.

[23] Hager, A.W. and Robertson, L., *Representing and ringifying a Riesz space*, Sympos. Math 21 (1977), 411-431.

[24] Hager, A.W. and Robertson, L., *Extremal units in an archimedean Riesz space*, Rendiconti del Seminario matematico della Università di Padova 59 (1978), 97-115.

[25] Henriksen, M., Vermeer, J., and Woods, R., *Wallman covers of compact spaces*, Instytut Matematyczny Polskiej Akademi Nauk (Warszawa), 1989.

[26] Herrlich, H. and Strecker, G., “Category Theory: an Introduction”, Heldermann Verlag, 1979.

[27] Martinez, J., *Hull classses of archimedean lattice-ordered groups with unit: A survey”*, in “Ordered Algebraic Structures”, Springer, 2002, 89-121.

[28] Martinez, J., *Polar functions*—II: *Completion classes of archimedean f-algebras vs. covers of compact spaces*, J. Pure Appl. Algebra 190(1-3) (2004), 225-249.

[29] Porter, R. and Woods G., “Extensions and absolutes of Hausdorff spaces”, Springer Science & Business Media, 2012.

[30] Woods, G., *Covering Properties and Coreflective Subcategories a, b*, Ann. New York Acad. Sci. 552(1) (1989), 173-184.

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