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DIAGONAL SUMS OF DOUBLY SUBSTOCHASTIC MATRICES*

LEI CAO†, ZHI CHEN‡, XUEFENG DUAN§, SELCUK KOYUNCU¶, AND HUILAN LI‖

Abstract. Let Ω_n denote the convex polytope of all $n \times n$ doubly stochastic matrices, and ω_n denote the convex polytope of all $n \times n$ doubly substochastic matrices. For a matrix $A \in \omega_n$, define the sub-defect of A to be the smallest integer k such that there exists an $(n+k) \times (n+k)$ doubly stochastic matrix containing A as a submatrix. Let $\omega_{n,k}$ denote the subset of ω_n which contains all doubly substochastic matrices with sub-defect k. For π a permutation of symmetric group of degree n, the sequence of elements $a_{1\pi(1)}, a_{2\pi(2)}, \ldots, a_{n\pi(n)}$ is called the diagonal of A corresponding to π. Let $h(A)$ and $l(A)$ denote the maximum and minimum diagonal sums of $A \in \omega_{n,k}$, respectively. In this paper, existing results of h and l functions are extended from Ω_n to $\omega_{n,k}$. In addition, an analogue of Sylvesters law of the h function on $\omega_{n,k}$ is proved.

Key words. Doubly substochastic matrices, Sub-defect, Maximum diagonal sum.

AMS subject classifications. 15A51, 15A83.

1. Introduction.

An n by n real matrix $A = [a_{ij}]$ is called a doubly stochastic matrix if

1. $a_{ij} \geq 0$, and
2. $\sum_i a_{ij} = 1$ and $\sum_j a_{ij} = 1$ for all i and j.

One can define doubly substochastic matrices by replacing the equalities by inequalities $\sum_i a_{ij} \leq 1$ and $\sum_j a_{ij} \leq 1$ in (2). Doubly stochastic matrices and doubly substochastic matrices have been studied intensively by many mathematicians (see [3], [7], [9] and [11]). Denote Ω_n and ω_n the set of all n by n doubly stochastic matrices and the set of all $n \times n$ doubly substochastic matrices, respectively. It is clear that $\Omega_n \subseteq \omega_n$. For $B \in \omega_n$, denote the sum of all elements of B by $\sigma(B)$, i.e

$$\sigma(B) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}. \tag{1.1}$$

Recently, Cao, Koyuncu and Parmer defined an interesting characteristic called sub-defect on the set ω_n. For $B \in \omega_n$, the sub-defect of B is denoted by $sd(B)$. It is the smallest integer k such that there exists an $(n+k) \times (n+k)$ doubly stochastic matrix containing B as a submatrix. It has been shown that the sub-defect can be calculated easily by taking the ceiling of the difference of the size of the matrix and the sum of all entries (see [4], [5] and [6]).

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Theorem 1.1. (Theorem 2.1 of [6]) *Let $B = [b_{ij}]$ be an $n \times n$ doubly substochastic matrix. Then*

$$sd(B) = \lceil n - \sigma(B) \rceil,$$

where $\lceil x \rceil$ is the ceiling of x.

Let $\omega_{n,k}$ denote the set of matrices in ω_n with sub-defect equal to k. It is worth to point out that the sub-defect k then provides a way to partition ω_n into $n+1$ convex subsets which are $\omega_{n,0} = \Omega_n, \omega_{n,1}, \ldots, \omega_{n,n}$. Namely,

(i) $\omega_{n,k}$ is convex for all k;
(ii) $\omega_{n,i} \cap \omega_{n,j} = \emptyset$ for $i \neq j$;
(iii) $\bigcup_{i=0}^{n} \omega_{n,i} = \omega_n$.

Let $A = [a_{ij}]$ be a real $n \times n$ matrix. Denote S_n the symmetric group of degree n. For $\pi \in S_n$, the sequence of elements $a_{1\pi(1)}, a_{2\pi(2)}, \ldots, a_{n\pi(n)}$ is called the diagonal of A corresponding to π and will also be denoted by π. A diagonal π of A is a maximum (minimum) diagonal if $\sum_{i=1}^{n} a_{i\pi(i)}$ is a maximum (minimum) among all $n!$ diagonal sums. The value of the maximum and minimum diagonal sums of A will be denoted by $h(A)$ and $l(A)$, respectively, and in case the matrix under consideration is fixed, simply by h and l, respectively. For $X = [x_{ij}]$ an $n \times n$ real matrix, denote

$$\langle A, X \rangle = \sum_{i,j} a_{ij} x_{ij}.$$

Note that $h(A)$ is also the support function of the assignment polytope Ω_n, i.e.,

$$h(A) = \sup\{\langle A, X \rangle : X \in \Omega_n\}.$$

Similarly, $l(A)$ can be defined as

$$l(A) = \inf\{\langle A, X \rangle : X \in \Omega_n\}.$$

In [12], Wang investigated and conjectured some interesting properties when the domains of these two functions are restricted on Ω_n. We extend the existing results of h function and l function on ω_n.

The paper is organized as follows: In Section 2, we show some properties of h-function and l-function on $\omega_{n,k}$ with respect to the sub-defect k. In Section 3, we prove an analogue of the Sylvesters law of h functions on $\omega_{n,k}$. In addition, we give an example to illustrate that the analogue of Frobenius inequalities of the rank function is not true on ω_n. Throughout this paper, we denote by J_n the $n \times n$ matrix whose all entries are 1.

2. The h-function and l-function on $\omega_{n,k}$. In this paper, we shall view h and l as two functions defined on $\omega_{n,k}$ in the natural way and study their properties. For $k = 0$, which is when restricted on Ω_n, some interesting properties have been discussed and explored in [12]. For $k \geq 1$, one crucial difference between matrices in Ω_n and those in $\omega_{n,k}$ is the sum of all elements. That is actually how sub-defect is defined originally. If $A \in \omega_{n,k}$, then $\sigma(A)$ is inside the interval $[n-k, n-k+1)$. We explore and show properties of the h and l functions on $\omega_{n,k}$ with respect to the sub-defect k or the sum of all elements of the matrices. We first notice that in $\omega_{n,k}$, the function h is convex while the function l is concave.

Proposition 2.1. *(i) h is a convex function;*
(ii) l is a concave function.

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Proof. Let A and B be two nonnegative matrices and $\lambda \in [0,1]$. It is clear that

$$h(\lambda A + (1-\lambda)B) \leq h(\lambda A) + h((1-\lambda)B) = \lambda h(A) + (1-\lambda)h(B)$$

and

$$l(\lambda A + (1-\lambda)B) \geq l(\lambda A) + l((1-\lambda)B) = \lambda l(A) + (1-\lambda)l(B),$$

and hence, the proposition holds. □

Let $A \in \omega_n$. It is not hard to see the extreme values of $h(A)$ and $l(A)$ given by the following proposition.

Proposition 2.2. *Let $A \in \omega_n$. Then*

$$0 \leq l(A) \leq \frac{\sigma(A)}{n} \leq h(A) \leq \sigma(A). \tag{2.2}$$

Proof. It is clear that $l(A) \geq 0$ and $h(A) \leq \sigma(A)$. From the covering theorem (Theorem 2.1 in [12]), we can get $l(A) \leq \frac{\sigma(A)}{n} \leq h(A)$, which implies the proposition. □

In (2.2), $l(A) = 0$ if and only if A has a zero diagonal, such as partial permutation matrices. On the other hand, $h(A) = \sigma(A)$ if and only if A has only one non-zero diagonal such that the sum of all entries of the diagonal is equal to $\sigma(A)$. For example, let $n-k \leq s < n-k+1$ and A an n by n matrix containing $\lfloor s \rfloor$ 1's and an $s - \lfloor s \rfloor$ on the diagonal as follows.

$$A = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & s-\lfloor s \rfloor & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix}.$$

It is easy to check that $h(A) = \sigma(A) = s$. For $A \in \omega_n$, denote $\sigma(A) = s$. Then $l(A) = \frac{s}{n} = h(A)$ if and only if A is in the following form:

$$A = \frac{s}{tn} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $t \geq s$, a positive integer and the first t rows of A are filled up by $\frac{s}{tn}$.

Corollary 2.3. *Let $B \in \omega_{n,k}$. Then*

$$\frac{n-k}{n} \leq h(B) < n-k+1.$$

Proof. This is a direct consequence of Proposition 2.2 and Theorem 1.1, which implies that $n-k \leq \sigma(B) < n-k+1$. □

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REMARK 2.4. From [12], we know that for $A \in \Omega_n$, $h(A) \geq 1$ with equality if and only if $A = \frac{1}{n} J_n$. However, in $\omega_{n,k}$, such an B satisfying $h(B) = \frac{n-k}{n}$ is not unique. For example, we can take $B_1 = \frac{n-k}{n^2} J_n$, and B_2 an n-square matrix with $n-k$ rows filled up by $\frac{1}{n}$'s, i.e.,

$$B_1 = \frac{n-k}{n^2} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad B_2 = \frac{1}{n} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By direct computation, we have $\sigma(B_1) = \sigma(B_2) = n-k$ and $h(B_1) = h(B_2) = \frac{n-k}{n}$.

Actually, if $A, B \in \omega_n$, then $AB \in \omega_n$ (Proposition 2.4 in [5]). We can evaluate the extreme values of $h(AB)$ and $l(AB)$.

THEOREM 2.5. *Let $A \in \omega_{n,k}$ and B be an $n \times n$ real matrix with nonnegative entries. Then*

(i) $h(AB) \leq h(B)$;
(ii) $l(B) \leq l(AB)$.

Proof. (i) For reader's convenience, we first prove a special case when $k = 0$, which means $A \in \Omega_n$. The case that both A and B in Ω_n has been proved in [12].

Due to Birkhoff's theorem (see [2], [3] and [10]), we can always write

$$A = \alpha_1 P_1 + \cdots + \alpha_m P_m,$$

where $P_1, \ldots, P_m$ are permutation matrices and $\alpha_1 + \cdots + \alpha_m = 1$. It is clear that $h(B) = h(PB)$ for an arbitrary permutation matrix P. Then we have

$$\begin{aligned} h(AB) &= h(\alpha_1 P_1 B + \cdots + \alpha_m P_m B) \\ &\leq \alpha_1 h(P_1 B) + \cdots + \alpha_m h(P_m B) \\ &= \alpha_1 h(B) + \cdots + \alpha_m h(B) \\ &= (\alpha_1 + \cdots + \alpha_m) h(B) \\ &= h(B) \end{aligned} \tag{2.3}$$

in which the inequality sign is due to the convexity of h.

Next we show the inequality holds for any integer $0 \leq k \leq n$ and all $A \in \omega_{n,k}$. Simply, let

$$\tilde{A} = \begin{bmatrix} A & X \\ Y & Z \end{bmatrix} \tag{2.4}$$

be a doubly stochastic matrix containing A as a principal submatrix. (For instance, we can let $\tilde{A}$ be the minimal doubly stochastic completion obtained by the method described in the proof of Theorem 2.1 in [6].) Write

$$\tilde{B} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \tag{2.5}$$

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with the same size as $\tilde{A}$. Since $\tilde{A}$ is a doubly stochastic matrix, we can apply (2.3) to $\tilde{A}$ and $\tilde{B}$ to get

$$h(AB) \le h\left(\begin{bmatrix} AB & 0 \\ YB & 0 \end{bmatrix}\right) = h(\tilde{A}\tilde{B}) \le h(\tilde{B}) = h(B).$$

(ii) For $A \in \Omega_n$, replacing h function by l function and using the concavity of l in (2.3), we get

$$l(AB) \ge l(B). \tag{2.6}$$

Then applying (2.6) to $\tilde{A}$ and $\tilde{B}$ defined in (2.4) and (2.5), respectively, we have

$$l(AB) \ge l\left(\begin{bmatrix} AB & 0 \\ YB & 0 \end{bmatrix}\right) = l(\tilde{A}\tilde{B}) \ge l(\tilde{B}) = l(B).$$ □

Corollary 2.6. *Let $A, B \in \omega_{n,k}$. Then*

(i) $h(AB) \le \min\{h(A), h(B)\}$;
(ii) $l(AB) \ge \max\{l(A), l(B)\}$.

Remark 2.7. To determine whether the equality in (i) holds, simply let $A = B = \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix}$. Then we have $AB = A = B$, and therefore, $h(AB) = h(A) = h(B) = \min\{h(A), h(B)\}$.

Remark 2.8. In [12], Wang shows that for $A, B \in \Omega_n$, $h(AB) \le h(A)h(B)$. However similar result does not hold for $A, B \in \omega_{n,k}$. To see this, simply choose

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \in \omega_{3,2}$$

and B just the transpose of A, i.e., $B = A^t$. Since

$$AB = \begin{bmatrix} \frac{3}{16} & \frac{3}{16} & 0 \\ \frac{3}{16} & \frac{3}{16} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we have $h(AB) = \frac{3}{8}$. However $h(A) = h(B) = \frac{1}{2}$, and hence,

$$\frac{3}{8} = h(AB) > h(A)h(B) = \frac{1}{4}.$$

Corollary 2.9. *Let $A \in \omega_{n,k}$. Then*

(i) $h(A^m) \le h(A)$;
(ii) $l(A^m) \ge l(A)$.

Lemma 2.10. *Let $A, B \in \omega_n$. Then*

$$0 \le l(AB) \le \frac{\sigma(A)\sigma(B)}{n^2} \le h(AB).$$

Proof. The leftmost inequality is trivial.

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To show

$$\frac{\sigma(A)\sigma(B)}{n^2} \leq h(AB),$$

let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices in ω_n. First note that $AB = [\sum_{k=1}^n a_{ik}b_{kj}]$. Without loss of generality, assume

$$h(AB) = \sum_{i,j=1}^n a_{ij}b_{ji}.$$

We need to find the minimum value of $h(AB)$ subject to the conditions

$$\sum_{i,j=1}^n a_{ij} = \sigma(A)$$

and

$$\sum_{i,j=1}^n b_{ij} = \sigma(B).$$

We introduce two Lagrange multipliers λ_1 and λ_2, and then construct the Lagrange function H as follows.

$$H = h(AB) - \lambda_1 \left(\sum_{i,j} a_{ij} - \sigma(A) \right) - \lambda_2 \left(\sum_{i,j} b_{ij} - \sigma(B) \right).$$

Using Lagrange multiplier method, we have

$$\frac{\partial H}{\partial a_{ij}} = b_{ji} - \lambda_1 = 0,$$
$$\frac{\partial H}{\partial b_{ij}} = a_{ji} - \lambda_2 = 0,$$
$$\frac{\partial H}{\partial \lambda_1} = \sum_{i,j} a_{ij} - \sigma(A) = 0,$$
$$\frac{\partial H}{\partial \lambda_2} = \sum_{i,j} b_{ij} - \sigma(B) = 0.$$

Solving the system of equations above, we get

$$\sum_{i,j} a_{ij} = n^2\lambda_2 = \sigma(A), \quad \sum_{i,j} b_{ij} = n^2\lambda_1 = \sigma(B),$$
$$a_{ij} = \lambda_2 = \frac{\sigma(A)}{n^2}, \quad b_{ij} = \lambda_1 = \frac{\sigma(B)}{n^2}.$$

Due to the convexity of the function h, we know that

$$h_{min}(AB) = \frac{\sigma(A)\sigma(B)}{n^2}.$$

Similarly, by the method of Lagrange multipliers and the concavity of l function, we can prove that

$$l_{max}(AB) = \frac{\sigma(A)\sigma(B)}{n^2}. \qquad \square$$

Since both l function and h function are well defined on the set of all $n \times n$ real matrices, although $A + B$ is not necessarily in ω_n for $A, B \in \omega_{n,k}$, both $l(A + B)$ and $h(A + B)$ are well defined and we have the following result.

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Proposition 2.11. *Let $A, B \in \omega_{n,k}$. Then*

(i) $0 \le h(A) + h(B) - h(A+B) \le \min\{h(A), h(B)\} < n - k + 1$;

(ii) $l(A+B) - l(A) - l(B) < \dfrac{2(n-k+1)}{n}$.

Proof. (i) Since $h(A+B) \le h(A) + h(B)$, it is clear that

$$0 \le h(A) + h(B) - h(A+B) \le \min\{h(A), h(B)\} < n - k + 1,$$

where the equality implies $h(A+B) = \max\{h(A), h(B)\}$. To see the upper bound is sharp, one can choose such A and B that both contain $n-k$ 1's and an ϵ as follows:

$$A = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \epsilon & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & 0 & 0 & & \\ & & & & & \ddots & 0 \\ \epsilon & & & & & & 0 \end{bmatrix},$$

where $0 \le \epsilon < 1$. Then $h(A) = h(B) = h(A+B) = n - k + \epsilon$, letting $\epsilon \to 1$ and we get $\sup_{A,B\in\omega_{n,k}}\{h(A) + h(B) - h(A+B)\} = n - k + 1$.

(ii) Since $\frac{1}{2}(A+B) \in \omega_{n,k}$, we have $l(\frac{A+B}{2}) < \frac{n-k+1}{n}$ or $l(A+B) < \frac{2(n-k+1)}{n}$. With $l(A), l(B) \ge 0$, we get $l(A+B) - l(A) - l(B) < \frac{2(n-k+1)}{n}$. □

3. The analogue of the Sylvesters law of the maximum diagonals of matrices in $\omega_{n,k}$. The Sylverster's law of the rank function (2.17.8 in [8]) says that if A is an $m \times t$ real matrix and B an $t \times n$ real matrix, then

$$\max\{\text{rank}(A), \text{rank}(B)\} \le \text{rank}(A) + \text{rank}(B) - \text{rank}(AB) \le n.$$

In [12], Wang conjectured the analogue of Sylvester's law of h function on Ω_n, and later on it was proved by Balasubramanian for a more general case using the statement $\text{tr}(A) + \text{tr}(B) - \text{tr}(AB) \le n$. For further use, we state the result as follows.

Theorem 3.1. (Main Theorem of [1]) *If A, B are $n \times n$ real matrices with all elements in the closed interval $[0, 1]$, then*

$$h(A) + h(B) - h(AB) \le n. \tag{3.7}$$

Also, Balasubramanian gave the conditions for which the equality holds. Based on this theorem, we give two analogues of (3.7) as follows.

Lemma 3.2. *Let $A \in \Omega_n$ and $B \in \omega_n$. Then*

$$1 \le h(A) + h(B) - h(AB) \le n,$$

where both the equalities can be tight.

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Proof. The inequality involving the upper bound is due to Theorem 3.1. To get the equality of the upper bound, simply take A to be any permutation matrix and any $B \in \omega_n$. In this case, $h(AB) = h(B)$, and therefore, $h(A) + h(B) - h(AB) = h(A) = n$.

For the lower bound, it is due to the combination of $h(A) \geq 1$ and Theorem 2.5 (i). Thus, we have

$$h(A) + h(B) - h(AB) \geq h(A) \geq 1.$$

The equality for the lower bound holds when $A = \frac{1}{n} J_n \in \Omega_n$ and $B = \frac{s}{n^2} J_n \in \omega_n$, where $0 < s < n$. In this case, $AB = B$ and then $h(B) = h(AB) = \frac{s}{n}$, which implies that

$$h(A) + h(B) - h(AB) = h(A) = 1. \qquad \square$$

Let $A, B \in \omega_{n,k}$. Then, due to Theorem 3.1 and Corollary 2.6 (i), we have

$$\max\{h(A), h(B)\} \leq h(A) + h(B) - h(AB) \leq n. \tag{3.8}$$

When $k = 0$, i.e., $A, B \in \Omega_n$, both upper bound and lower bound are tight. However, when k is close to n, the upper bound is not tight anymore. In addition, it seems that the lower bound can be more precise with respect to the sub-defect k. So, we explore the role of k and obtain the following theorem for the doubly substochastic matrix case, which is stronger than (3.8).

THEOREM 3.3. *Let $A, B \in \omega_{n,k}$. Then*

$$\frac{n-k}{n} \leq h(A) + h(B) - h(AB) \leq \min\{n, 2(n-k+1)\}.$$

In particular when $k \geq \frac{n}{2} + 1$,

$$\sup_{A,B \in \omega_{n,k}} \{h(A) + h(B) - h(AB)\} = 2(n-k+1).$$

In order to prove Theorem 3.3, we need the following lemma.

LEMMA 3.4. *Let $A \in \omega_{n,k}$. Then we have $h(A) < n-k+1$ and*

$$\sup_{A \in \omega_{n,k}} \{h(A)\} = n-k+1.$$

Proof. Since $A \in \omega_{n,k}$, $\sigma(A) < n-k+1$. It is clear that $h(A) \leq \sigma(A) < n-k+1$. So, $n-k+1$ is an upper bound. To show $n-k+1$ is the least upper bound, one can construct the following diagonal matrix:

$$A_\epsilon = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \epsilon & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix} \tag{3.9}$$

which contains $n-k$ 1's and an ϵ on the diagonal. For $0 \leq \epsilon < 1$, $A_\epsilon \in \omega_{n,k}$. Note that

$$\lim_{\epsilon \to 1^-} h(A_\epsilon) = n-k+1,$$

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which means that

$$\sup_{A\in\omega_{n,k}} \{h(A)\} = n-k+1. \qquad \square$$

COROLLARY 3.5. *Let $A\in\omega_{n,k}$. Then there exists an $0\le\epsilon<1$, such that*

$$h(A)\le h(A_\epsilon).$$

Proof. It is clear that

$$h(A_\epsilon)=\max\{h(A):\sigma(A)=\sigma(A_\epsilon), A\in\omega_{n,k}\}.$$

Therefore, the corollary holds. $\square$

Now, we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. Upper bound. On the one hand, due to Theorem 3.1, A and B satisfy

$$h(A)+h(B)-h(AB)\le n.$$

Since when $0\le k<\frac{n}{2}+1$ we have $2(n-k+1)>n$, and therefore, the right hand side inequality in Theorem 3.3 holds. For $k\ge\frac{n}{2}+1$, we have

$$2(n-k+1)\le 2\left(n-(\frac{n}{2}+1)+1\right)=n.$$

Thus, we need to show that when $k\ge\frac{n}{2}+1$, $h(A)+h(B)-h(AB)\le 2(n-k+1)$. To see this, let A_ϵ be as in (3.9) and B_η be the matrix as follows.

$$B_\eta=\begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & \eta & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

which contains $n-k$ 1's and a nonnegative real number $0\le\eta<1$. Since $k\ge\frac{n}{2}+1$, $A_\epsilon B_\eta=\mathbf{0}$, and hence, $h(A_\epsilon B_\eta)=0$. In addition, due to Corollary 3.5, we have both

$$\max_{A\in\omega_{n,k}} h(A)\le\lim_{\epsilon\to1^-} h(A_\epsilon)=n-k+1,$$

and

$$\max_{B\in\omega_{n,k}} h(B)\le\lim_{\eta\to1^-} h(B_\eta)=n-k+1.$$

Therefore, we claim that

$$h(A)+h(B)-h(AB)\le\lim_{\epsilon\to1^-} h(A_\epsilon)+\lim_{\eta\to1^-} h(B_\eta)=2(n-k+1).$$

Lower bound. Due to Corollary 2.3 and Corollary 2.6, we have

$$h(A)+h(B)-h(AB)\ge\max\{h(A),h(B)\}\ge\frac{n-k}{n},$$

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which means that $\frac{n-k}{n}$ is a lower bound. It is tight because one can always let

$$A_0 = \frac{1}{n}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \omega_{n,k}$$

such that all elements in the the first $n-k$ rows are $\frac{1}{n}$ and 0 otherwise. Let $B_0 = A_0^t$. Then

$$A_0B_0 = \frac{1}{n}\begin{bmatrix} J_{n-k} & 0 \\ 0 & 0 \end{bmatrix}.$$

So, we have

$$h(A_0) = h(B_0) = h(A_0B_0) = \frac{n-k}{n},$$

and hence,

$$h(A_0) + h(B_0) - h(A_0B_0) = \frac{n-k}{n}. \qquad \square$$

In [12], the authors also conjectured the analogue of Frobenius inequalities of the rank function (see page 27 in [8]).

Conjecture 3.6. (Conjecture 5.2 of [12]) *Let $A, B, C \in \Omega_n$. Then*

$$h(AB) + h(BC) - h(ABC) \leq h(B).$$

Note that (3.7) is a special case of Conjecture 3.6 by letting B be the identity matrix. Although the Sylvester's law of h function is true and Conjecture 3.6 still remains mysterious to us, it is not true if we replace Ω_n by ω_n in Conjecture 3.6. Here is an example.

Example 3.7. Let

$$A = \frac{1}{5}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$B = A^t$ and $C = A$. Then

$$AB = \frac{1}{5}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad BC = \frac{3}{25}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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and

$$ABC = \frac{3}{25}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, $h(A) = h(B) = h(AB) = h(BC) = \frac{3}{5}$ and $h(ABC) = \frac{9}{25}$, and hence,

$$\frac{21}{25} = h(AB) + h(BC) - h(ABC) > h(B) = \frac{3}{5}.$$

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