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Spectral Properties of a Sequence of Matrices Connected to Each Other via Schur Complement and Arising in a Compartmental Model

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Spectral properties of a sequence of matrices connected to each other via Schur complement and arising in a compartmental model

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Abstract: We consider a sequence of real matrices A_n which is characterized by the rule that A_{n-1} is the Schur complement in A_n of the (1,1) entry of A_n, namely $-e_n$, where e_n is a positive real number. This sequence is closely related to linear compartmental ordinary differential equations. We study the spectrum of A_n. In particular, we show that A_n has a unique positive eigenvalue λ_n and $\{\lambda_n\}$ is a decreasing convergent sequence. We also study the stability of A_n for small n using the Routh-Hurwitz criterion.

Keywords: Schur complement; Routh-Hurwitz criterion, elementary symmetric polynomials; linear compartmental model; latency phase

MSC: 15A18; 15B99

1 Introduction

The Schur complement of a matrix provides a method of relating matrices of different dimension with similar structure. Interestingly, we find that, when performing the Schur complement with a block that consists of a single negative real number, the resulting matrix is the same as the coefficient matrix for a linear compartmental ordinary differential equation model when a latency phase is added to precede the first compartment and the last compartment feeds back to the first compartment. Here we study the spectral properties of a sequence of matrices resulting from the aforementioned Schur-complement based construction as they relate to the study of the (asymptotic) stability of a linear compartmental model.

2 Background

Let $M_n(\mathbb{R})$ denote the space of $n \times n$ matrices with real entries and let $A \in M_2(\mathbb{R})$ be the matrix

$$A = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix},$$

where $a, b, c, d > 0$ and $\det(A) = ad - bc < 0$. A has one positive and one negative eigenvalue. Consider any sequence $\{e_n\}_{n\geq 3}$ of positive real numbers. Starting the terms of this sequence from $n = 3$ is just a matter of

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notation. We are going to study a sequence of matrices $A_n \in M_n(\mathbb{R})$ such that $(A_n)_{1,1} = -e_n$ and A_{n-1} is the Schur complement of $(A_n)_{1,1}$ in A_n.

Definition 1. *(Schur Complement, [1]) Let* $M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ *be a block matrix where X is invertible. The Schur Complement of block X in matrix M is defined as*

$$M/X = W - ZX^{-1}Y.$$

We also have

$$det(M) = det(X)det(W - ZX^{-1}Y).$$

We construct the sequence A_n as follows: using the matrix A given above and any sequence of positive real numbers $\{e_3, e_4, \dots\}$, we set

$$A_3 = \begin{pmatrix} -e_3 & 0 & b \\ e_3 & -a & 0 \\ 0 & c & -d \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where $X = (-e_3)$, $Y = \begin{pmatrix} 0 & b \end{pmatrix}$, $Z = \begin{pmatrix} e_3 \\ 0 \end{pmatrix}$, and $W = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$. Thus, we have

$$A_3/X = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix} + \frac{1}{e_3}\begin{pmatrix} e_3 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & b \end{pmatrix} = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix} = A.$$

Next, we construct

$$A_4 = \begin{pmatrix} -e_4 & 0 & 0 & b \\ e_4 & -e_3 & 0 & 0 \\ 0 & e_3 & -a & 0 \\ 0 & 0 & c & -d \end{pmatrix},$$

and we note that $A_3 = A_4/(-e_4)$. Continuing the same way, we obtain the nth matrix

$$A_n = \begin{pmatrix} -e_n & 0 & \dots & \dots & \dots & 0 & b \\ e_n & -e_{n-1} & \ddots & & & & 0 \\ 0 & e_{n-1} & -e_{n-2} & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & e_4 & -e_3 & \ddots & \vdots \\ \vdots & & & \ddots & e_3 & -a & 0 \\ 0 & \dots & \dots & \dots & 0 & c & -d \end{pmatrix},$$

and $A_{n-1} = A_n/(-e_n)$.

Example 1. *Consider the matrix*

$$A = \begin{pmatrix} -0.2 & 12 \\ 10 & -0.1 \end{pmatrix}$$

and the Fibonacci sequence $\{e_3 = 1, e_4 = 1, e_5 = 2, \ldots\}$. *The first three terms of the sequence of matrices* A_n *are given by*

$$A_3 = \begin{pmatrix} -1 & 0 & 12 \\ 1 & -0.2 & 0 \\ 0 & 10 & -0.1 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} -1 & 0 & 0 & 12 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -0.2 & 0 \\ 0 & 0 & 10 & -0.1 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} -2 & 0 & 0 & 0 & 12 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -0.2 & 0 \\ 0 & 0 & 0 & 10 & -0.1 \end{pmatrix}.$$

Ordinary differential equation systems arising from compartmental modeling are ubiquitous in mathematical modeling (see for example [2] or [3] and references therein). For a viral infection compartmental model, the study of local stability involves the linear system $\frac{d\vec{x}}{dt} = A\vec{x}$ with A as defined above [3]. The entries of A as given above represent transition rates between the compartments represented by the entries in $\vec{x}$. The values a and c represent a feed forward process and give the rates per unit time for exiting compartment x_2 and entering compartment x_1, respectively. Similarly, the values d and b represent a feedback process and give the rates per unit time for exiting compartment x_1 and entering compartment x_2, respectively. In the viral infection compartmental model example, x_1 and x_2 represent densities of virons and infected cells, respectively. These models can be easily nested inside models with an ‘eclipse’ or ‘latency’ phase [3] and the corresponding linearized model would take the form $\frac{d\vec{x}_3}{dt} = A_3\vec{x}_3$ where $\vec{x}_3 = \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix}$. The new compartment x_3 serves to delay the process of data transitioning to compartment x_1 from the initial compartment. In this feed forward process, e_3 is the rate of transmission from compartment x_3 to compartment x_2. The resultant zero column sum indicates no gain or loss from compartment x_3. Continuing to add latency phases with feedback from the last compartment to the first results in the sequence of matrices defined above. Note that the solutions of the system $\frac{d\vec{x}_n}{dt} = A_n\vec{x}_n$ are given by $\vec{x}_n(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}\vec{v}_i$, where λ_i and $\vec{v}_i$ are the eigenvalues and eigenvectors of A_n and c_i are constants chosen to match the initial conditions of the system.

3 Preliminary results

Since A_{n-1} is the Schur complement of $(A_n)_{1,1} = -e_n$ in A_n, it follows from Definition 1 that $\det(A_n) = -e_n\det(A_{n-1})$. Hence, the matrices A_n are all invertible. Moreover, we have

$$\begin{aligned} \det(A_3) &= (-e_3)\det(A) = -e_3(ad - bc) = (-1)^4 e_3(bc - ad) > 0 \\ \det(A_4) &= (-e_4)\det(A_3) = -e_3e_4(bc - ad) = (-1)^5 e_3e_4(bc - ad) < 0 \\ &\vdots \\ \det(A_n) &= (-e_n)\det(A_{n-1}) = (-1)^{n+1} e_3e_4\cdots e_n(bc - ad). \end{aligned}$$

We have proved the following lemma.

Lemma 1. *The matrices A_n are all invertible and we can write*

$$det(A_n) = (-1)^{n+1} e_3 \cdots e_n (bc - ad) = (-1)^n \Big(\prod_{k=3}^{n} e_k\Big) det(A). \quad (1)$$

Lemma 2. *For each $n \geq 3$, A_n has a unique positive eigenvalue of multiplicity one.*
Proof: We prove the lemma by deriving a general formula for the characteristic polynomial of A_n. First, note that the characteristic polynomials $\det(xI - A_3)$ and $\det(xI - A_4)$ are given by

$$\begin{aligned} \det(xI - A_3) &= (x + e_3)(x + a)(x + d) - bce_3, \\ \det(xI - A_4) &= (x + e_4)(x + e_3)(x + a)(x + d) - bce_3e_4. \end{aligned}$$

Expanding these expressions, we obtain

$$\begin{aligned} \det(xI - A_3) &= x^3 + \sigma_1(a, d, e_3)x^2 + \sigma_2(a, d, e_3)x + (-1)^3 \det(A_3), \\ \det(xI - A_4) &= x^4 + \sigma_1(a, d, e_3, e_4)x^3 + \sigma_2(a, d, e_3, e_4)x^2 \\ &\quad + \sigma_3(a, d, e_3, e_4)x + (-1)^4 \det(A_4). \end{aligned}$$

where

$$\sigma_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$$

is the kth elementary symmetric polynomial in the alphabet $x_1, \dots x_n$. Continuing this way, we see that the characteristic polynomial $f_n(x) = \det(xI - A_n)$ of A_n can be written as

$$f_n(x) = x^n + \sum_{i=1}^{n-1} \sigma_i(a, d, e_3, \dots, e_n)x^{n-i} + (-1)^n \det(A_n) \quad (2)$$

for all $n \geq 3$. Since a, b, c, d and e_n for $n \geq 3$ are all positive, $f_n(x)$ has all but one positive real coefficients: the only negative coefficient is $(-1)^n \det(A_n)$ by Lemma 1. This implies that (for instance, by Descartes' rule of signs) $f_n(x)$ has exactly one positive root.

We also observe that the unique positive eigenvalue of A_n has multiplicity one. Indeed, it is evident from (2) that the first derivative $f_n'(x)$ is positive on $[0, \infty)$. In particular, $f_n'(x)$ cannot have a positive root. □

Remark 1 Let $f_n(x) = \det(xI - A_n)$ be the characteristic polynomial of A_n. As we have seen in the proof of Lemma 2, we have $f_n'(x) > 0$ for $x \in [0, \infty)$. In particular, $f_n(x)$ is increasing over $[0, \infty)$.

4 Eigenvalues, main theorem and stability

We will denote the unique positive eigenvalue of A_n by λ_n for $n \geq 3$. We show that the sequence $\{\lambda_n\}$ is a decreasing convergent sequence.

Lemma 3. *The characteristic polynomial $f_n(x)$ of A_n admits the following expression:*

$$f_n(x) = (x + e_n)f_{n-1}(x) + \sigma_{n-1}(b, c, e_3, \dots, e_{n-1})x. \quad (3)$$

Proof: Recall that the characteristic polynomial of A_n is given by

$$f_n(x) = x^n + \sum_{i=1}^{n-1} \sigma_i(a, d, e_3, \dots, e_n)x^{n-i} + (-1)^n \det(A_n).$$

We can write

$$\sigma_i(a, d, e_3, \dots, e_n) = \sigma_i(a, d, e_3, \dots, e_{n-1}) + e_n \sigma_{i-1}(a, d, e_3, \dots, e_{n-1})$$

and

$$\det(A_n) = (-1)^{n+1}\big(\sigma_n(b, c, e_3, \dots, e_n) - \sigma_n(a, d, e_3, \dots, e_n)\big).$$

Now, for simplicity in the notation we write $\sigma_i = \sigma_i(a, d, e_3, \dots, e_{n-1})$ and $\sigma_0 = 1$. Using the observations above we have

$$\begin{aligned} f_n(x) &= x^n + \sum_{i=1}^{n-1} \sigma_i x^{n-i} + e_n \sum_{i=1}^{n-1} \sigma_{i-1} x^{n-i} + (-1)^n \det(A_n) \\ &= x^n + \sum_{i=1}^{n-1} \sigma_i x^{n-i} + e_n \sum_{i=1}^{n-1} \sigma_{i-1} x^{n-i} + (-1)^{n-1} e_n \det(A_{n-1}) \\ &= x^n + \sum_{i=1}^{n-1} \sigma_i x^{n-i} + e_n \Big(\sum_{i=1}^{n-1} \sigma_{i-1} x^{n-i} + (-1)^{n-1} \det(A_{n-1})\Big) \\ &= x^n + \sum_{i=1}^{n-1} \sigma_i x^{n-i} + e_n \Big(\sum_{i=0}^{n-2} \sigma_i x^{n-i-1} + (-1)^{n-1} \det(A_{n-1})\Big) \\ &= x^n + \sum_{i=1}^{n-1} \sigma_i x^{n-i} + e_n f_{n-1}(x). \end{aligned}$$

By (2),

$$f_{n-1}(x) = x^{n-1} + \sum_{i=1}^{n-2} \sigma_i x^{n-i-1} + (-1)^{n-1} \det(A_{n-1}).$$

Multiplying both sides of the above equation by x, we obtain

$$x f_{n-1}(x) = x^n + \sum_{i=1}^{n-2} \sigma_i x^{n-i} + (-1)^{n-1} \det(A_{n-1}) x.$$

Thus,

$$\begin{aligned} f_n(x) &= x^n + \sum_{i=1}^{n-2} \sigma_i x^{n-i} + \sigma_{n-1} x + e_n f_{n-1}(x) \\ &= x f_{n-1}(x) - (-1)^{n-1} \det(A_{n-1}) x + \sigma_{n-1} x + e_n f_{n-1}(x) \\ &= (x + e_n) f_{n-1}(x) + ((-1)^n \det(A_{n-1}) + \sigma_{n-1}) x. \end{aligned}$$

Since $(-1)^n \det(A_{n-1}) = (-1)^n (-1)^n \big(\sigma_{n-1}(b, c, \dots, e_{n-1}) - \sigma_{n-1}\big)$, we finally obtain $f_n(x) = (x + e_n) f_{n-1}(x) + \sigma_{n-1}(b, c, e_3, \dots, e_{n-1}) x$. □

Lemma 4. *Let $\tilde{\lambda_n}$ be any eigenvalue of A_n such that $\tilde{\lambda_n} \neq \lambda_n$. Then $|\tilde{\lambda_n}| \geq \lambda_n$.*

Proof. Let's assume $|\tilde{\lambda_n}| < \lambda_n$. Since we know from Remark 1 that $f_n(x)$ is increasing on $[0, \infty)$, we must have $f_n(|\tilde{\lambda_n}|) < f_n(\lambda_n) = 0$. However, using (2) and setting $\sigma_i = \sigma_i(a, d, e_3, \dots, e_{n-1}, e_n)$ we obtain

$$\begin{aligned} f_n(|\tilde{\lambda_n}|) &= |\tilde{\lambda_n}|^n + \sum_{i=1}^{n-i} \sigma_i |\tilde{\lambda_n}|^{n-i} + (-1)^n \det(A_n) \\ &\geq \Big|(\tilde{\lambda_n})^n + \sum_{i=1}^{n-1} \sigma_i (\tilde{\lambda_n})^{n-i}\Big| + (-1)^n \det(A_n) \\ &\geq |f_n(\tilde{\lambda_n}) - (-1)^n \det(A_n)| + (-1)^n \det(A_n) \\ &= |(-1)^n \det(A_n)| + (-1)^n \det(A_n) = 0, \end{aligned}$$

because $(-1)^n \det(A_n) < 0$ for all n by Lemma 1. Therefore, we must have $|\tilde{\lambda_n}| \geq \lambda_n$.

□

Theorem 1. *The sequence $\{\lambda_n\}_{n\geq 3}$, with λ_n being the unique positive eigenvalue of A_n, is a decreasing convergent sequence.*

Proof. We know from Remark 1 that $f_n'(x) > 0$ on $[0, \infty)$. Let's assume that $\lambda_{n-1} < \lambda_n$ for some n. By the mean value theorem, there exists a point $p_n \in (\lambda_{n-1}, \lambda_n)$ such that

$$f_n'(p_n) = \frac{f_n(\lambda_n) - f_n(\lambda_{n-1})}{\lambda_n - \lambda_{n-1}} = \frac{-f_n(\lambda_{n-1})}{\lambda_n - \lambda_{n-1}},$$

where $f_n(\lambda_n) = 0$ since λ_n is an eigenvalue of A_n. On the other hand, by (3) we have $f_n(\lambda_{n-1}) = \sigma_{n-1}(b, c, e_3, \ldots, e_{n-1})\lambda_{n-1}$. Then,

$$f_n'(p_n) = -\frac{\sigma_{n-1}(b, c, e_3, \ldots, e_{n-1})\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} < 0.$$

But this is not possible because $f_n'(x) > 0$ on $(0, \infty)$. Therefore, $\{\lambda_n\}$ forms a decreasing sequence. Since the sequence $\{\lambda_n\}$ is decreasing and bounded from below by 0, it is a convergent sequence. □

Proposition 1. *Suppose that λ_n converges to $L \geq 0$ and the product $\prod_{k=3}^{n} e_k$ converges to 0. Then, $L < 1$ and there exists N such that $\lambda_n < 1$ for all $n \geq N$.*

Proof. If the product $\prod_{k=3}^{n} e_k$ converges to 0, then Lemma 1 indicates that

$$|\det(A_n)| = |\det(A)| \prod_{k=3}^{n} e_k \to 0.$$

We also have

$$|\det(A_n)| = \lambda_n \prod_j |\tilde{\lambda}_{n,j}|,$$

where the $\tilde{\lambda}_{n,j}$ are non-positive eigenvalues of A_n. From Lemma 4, we know that each $\tilde{\lambda}_{n,j}$ satisfies $|\tilde{\lambda}_{n,j}| \geq \lambda_n$. Moreover, since $\{\lambda_n\}$ is a decreasing sequence, $\lambda_n \geq L$. Based on these observations we have

$$|\det(A_n)| = \lambda_n \prod_j |\tilde{\lambda}_{n,j}| \geq L^n.$$

Therefore, since $|\det(A_n)| \to 0$, we must have $L^n \to 0$. This is the case if $L < 1$ as required.

It is not possible to have $\lambda_n \geq 1$ for all n. Otherwise, 1 would be a lower bound for the set $\{\lambda_n |\ n \geq 3\}$. But this contradicts the fact that $L < 1$. Therefore, there exists some N such that $\lambda_N < 1$. Because $\{\lambda_n\}$ is a decreasing sequence, we must have $\lambda_n < 1$ for all $n \geq N$ as claimed. □

Proposition 2. *The eigenvector of A_n corresponding to λ_n consists of all non-zero entries with the same sign.*

Proof. This is a consequence of the structure of A_n. When solving for the eigenvector, each component v_k of the eigenvector is a positive multiple of the prior component for $k = 2, 3, \ldots, n$ and v_1 is a positive multiple of v_n. If any component of the eigenvector were to be zero, then the vector would be the zero vector and thus it would fail to be an eigenvector. □

Recall that λ_n is the unique positive eigenvalue of A_n with multiplicity 1. Therefore, $f_n(x)$ is divisible by $(x - \lambda_n)$ and the quotient $g_n(x) = \dfrac{f_n(x)}{x - \lambda_n}$ is a degree $n-1$ monic polynomial which does not vanish at λ_n. All the remaining $n-1$ eigenvalues of A_n are the roots of $g_n(x)$. In the following theorem, we determine the location and structure of the roots of the characteristic polynomials $f_n(x)$ for small values of n. One can consult [4], [5], [6], [7], [8] for more information on various techniques to locate and characterize the roots of a polynomial.

Theorem 2. *All the eigenvalues of A_n except λ_n have negative real parts for $n = 3$ and $n = 4$.*

Proof. The characteristic polynomials of A_3 and A_4 are given by

$$\begin{aligned} f_3(x) &= x^3 + \sigma_1(a,d,e_3)x^2 + \sigma_2(a,d,e_3)x - \det(A_3) \\ f_4(x) &= x^4 + \sigma_1(a,d,e_3,e_4)x^3 + \sigma_2(a,d,e_3,e_4)x^2 + \sigma_3(a,d,e_3,e_4)x + \det(A_4). \end{aligned}$$

By applying polynomial division we obtain

$$\begin{aligned} g_3(x) &= x^2 + \big(\sigma_1(a,d,e_3) + \lambda_3\big)x + \big(\sigma_2(a,d,e_3) + \sigma_1(a,d,e_3)\lambda_3 + \lambda_3^2\big) \\ g_4(x) &= x^3 + a_1x^2 + a_2x + a_3, \end{aligned}$$

where

$$\begin{aligned} a_1 &= \sigma_1(a,d,e_3,e_4) + \lambda_4 \\ a_2 &= \sigma_2(a,d,e_3,e_4) + \sigma_1(a,d,e_3,e_4)\lambda_4 + \lambda_4^2 \\ a_3 &= \sigma_3(a,d,e_3,e_4) + \sigma_2(a,d,e_3,e_4)\lambda_4 + \sigma_1(a,d,e_3,e_4)\lambda_4^2 + \lambda_4^3. \end{aligned}$$

Note that all the coefficients of $g_3(x)$ and $g_4(x)$ are positive. To show that all roots of $g_3(x)$ and $g_4(x)$ have negative real parts, we will apply Routh-Hurwitz criterion [2].
Case $n = 3$
In this case it is sufficient to note $\sigma_1(a,d,e_3) + \lambda_3 > 0$ and $(\sigma_2(a,d,e_3) + \sigma_1(a,d,e_3)\lambda_3 + \lambda_3^2) > 0$.
Case $n = 4$
This case is less trivial. By Routh-Hurwitz criterion it is sufficient to show that $a_1 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$. The positivity of a_1 and a_3 is obvious. Now, for simplicity in the notation we will write $\sigma_i = \sigma_i(a,d,e_3,e_4)$ and $\lambda_4 = \lambda$. Then,

$$\begin{aligned} a_1a_2 - a_3 &= (\sigma_1 + \lambda)(\sigma_2 + \sigma_1\lambda + \lambda^2) - (\sigma_3 + \sigma_2\lambda + \sigma_1\lambda^2 + \lambda^3) \\ &= \sigma_1\sigma_2 - \sigma_3 + \sigma_1^2\lambda + \sigma_1\lambda^2. \end{aligned}$$

By definition, $\sigma_1 = a + d + e_3 + e_4$, $\sigma_2 = ad + ae_3 + ae_4 + de_3 + de_4 + e_3e_4$ and $\sigma_3 = ade_3 + ade_4 + ae_3e_4 + de_3e_4$. Then, we have

$$\sigma_1\sigma_2 = ade_3 + ade_4 + ae_3e_4 + de_3e_4 + \Theta = \sigma_3 + \Theta,$$

where Θ is positive. Therefore, $\sigma_1\sigma_2 - \sigma_3 = \Theta$ and so $a_1a_2 - a_3 > 0$, as required. □

Note that the number of roots of $g_3(x)$ and $g_4(x)$ having negative real part does not depend on the choice of the initial matrix A or the choice of the sequence $\{e_n\}$. It is curious to ask if the same can be said for $n > 4$. Unfortunately, this is not true for arbitrary A. For instance, in Example 1 we have

$$A_5 = \begin{pmatrix} -2 & 0 & 0 & 0 & 12 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -0.2 & 0 \\ 0 & 0 & 0 & 10 & -0.1 \end{pmatrix}$$

and $0.0927315107187 \pm 2.77510340056i$ are two complex eigenvalues of A_5 with positive real parts.

5 Discussion and further work

For a compartmental model as described in the Section 2, an interesting question is if a memory of non-zero initial data for the first compartment will appear in the last compartment as an increasing number of latency phases are added to the system. The results here, specifically the fact that the eigenvector corresponding to the unique positive real eigenvalue has all non-zero components, indicate that, regardless of number of phases, this memory will appear in the last compartment.

Although it is not true that all the eigenvalues (except the unique positive one) have negative real parts for all n, it is interesting to ask if it is possible to acquire this situation by imposing some restrictions on either the matrix A or the sequence $\{e_n\}$. Based on several simulations we have the following conjecture.

Conjecture If the product $\prod_{k=3}^{n} e_k$ is sufficiently small and the positive eigenvalue of the matrix A is less than 1, then all the eigenvalues of A_n (except the unique positive one) have negative real part for any n.

Indeed, the determinants that arise from application of the Routh-Hurwitz criterion involve several terms, some of which are negative. When rewriting these determinants as the sum of positive and negative parts, the negative parts can be written as a function of the unique positive eigenvalue λ_n, and the numbers $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ defined in the proof of Lemma 2. Our computations using a computer algebra system indicate that, under the conditions of the conjecture, the positive part of the determinant will be greater than the absolute value of the negative part making all such determinants positive as required by the Routh-Hurwitz criterion.

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