1-2017

Spectral Properties of a Sequence of Matrices Connected to Each Other via Schur Complement and Arising in a Compartmental Model

Evan Haskell  
*Nova Southeastern University*, haskell@nova.edu

Vehbi Emrah Paksoy  
*Nova Southeastern University*, vp80@nova.edu

Follow this and additional works at: [http://nsuworks.nova.edu/math_facarticles](http://nsuworks.nova.edu/math_facarticles)

Part of the [Mathematics Commons](http://nsuworks.nova.edu/math_facarticles)

NSUWorks Citation
[http://nsuworks.nova.edu/math_facarticles/206](http://nsuworks.nova.edu/math_facarticles/206)

This Article is brought to you for free and open access by the Department of Mathematics at NSUWorks. It has been accepted for inclusion in Mathematics Faculty Articles by an authorized administrator of NSUWorks. For more information, please contact nsuworks@nova.edu.
Spectral properties of a sequence of matrices connected to each other via Schur complement and arising in a compartmental model

Abstract: We consider a sequence of real matrices $A_n$ which is characterized by the rule that $A_{n-1}$ is the Schur complement in $A_n$ of the (1,1) entry of $A_n$, namely $-e_n$, where $e_n$ is a positive real number. This sequence is closely related to linear compartmental ordinary differential equations. We study the spectrum of $A_n$. In particular, we show that $A_n$ has a unique positive eigenvalue $\lambda_n$ and $\{\lambda_n\}$ is a decreasing convergent sequence. We also study the stability of $A_n$ for small $n$ using the Routh-Hurwitz criterion.

Keywords: Schur complement; Routh-Hurwitz criterion, elementary symmetric polynomials; linear compartmental model; latency phase

MSC: 15A18; 15B99

1 Introduction

The Schur complement of a matrix provides a method of relating matrices of different dimension with similar structure. Interestingly, we find that, when performing the Schur complement with a block that consists of a single negative real number, the resulting matrix is the same as the coefficient matrix for a linear compartmental ordinary differential equation model when a latency phase is added to precede the first compartment and the last compartment feeds back to the first compartment. Here we study the spectral properties of a sequence of matrices resulting from the aforementioned Schur-complement based construction as they relate to the study of the (asymptotic) stability of a linear compartmental model.

2 Background

Let $M_n(\mathbb{R})$ denote the space of $n \times n$ matrices with real entries and let $A \in M_2(\mathbb{R})$ be the matrix

$$A = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix},$$

where $a, b, c, d > 0$ and $\det(A) = ad - bc < 0$. $A$ has one positive and one negative eigenvalue. Consider any sequence $\{e_n\}_{n=3}$ of positive real numbers. Starting the terms of this sequence from $n = 3$ is just a matter of
notation. We are going to study a sequence of matrices $A_n \in M_n(\mathbb{R})$ such that $(A_n)_{1,1} = -e_n$ and $A_{n-1}$ is the Schur complement of $(A_n)_{1,1}$ in $A_n$.

**Definition 1.** (Schur Complement, [1]) Let $M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ be a block matrix where $X$ is invertible. The Schur Complement of block $X$ in matrix $M$ is defined as

$$M/X = W - ZX^{-1}Y.$$ 

We also have

$$\det(M) = \det(X)\det(W - ZX^{-1}Y).$$

We construct the sequence $A_n$ as follows: using the matrix $A$ given above and any sequence of positive real numbers $\{e_3, e_4, \ldots\}$, we set

$$A_3 = \begin{pmatrix} -e_3 & 0 & b \\ e_3 & -a & 0 \\ 0 & c & -d \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where $X = (-e_3)$, $Y = \begin{pmatrix} 0 \\ b \end{pmatrix}$, $Z = \begin{pmatrix} e_3 \\ 0 \end{pmatrix}$, and $W = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$. Thus, we have

$$A_3/X = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix} + \frac{1}{e_3} \begin{pmatrix} e_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix} = A.$$ 

Next, we construct

$$A_4 = \begin{pmatrix} -e_4 & 0 & 0 & b \\ e_4 & -e_3 & 0 & 0 \\ 0 & e_3 & -a & 0 \\ 0 & 0 & c & -d \end{pmatrix},$$

and we note that $A_3 = A_4/(-e_4)$. Continuing the same way, we obtain the $n$th matrix

$$A_n = \begin{pmatrix} -e_n & 0 & \ldots & \ldots & \ldots & 0 & b \\ e_n & -e_{n-1} & \ddots & \vdots & \vdots & \vdots & 0 \\ 0 & e_{n-1} & -e_{n-2} & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & e_4 & -e_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & e_3 & -a & 0 \\ 0 & \ldots & \ldots & \ldots & 0 & c & -d \end{pmatrix},$$

and $A_{n-1} = A_n/(-e_n)$.

**Example 1.** Consider the matrix

$$A = \begin{pmatrix} -0.2 & 12 \\ 10 & -0.1 \end{pmatrix}$$
and the Fibonacci sequence \( \{ e_3 = 1, e_4 = 1, e_5 = 2, \ldots \} \). The first three terms of the sequence of matrices \( A_n \) are given by

\[
\begin{align*}
A_3 &= \begin{pmatrix} -1 & 0 & 12 \\ 1 & -0.2 & 0 \\ 0 & 10 & -0.1 \end{pmatrix}, \\
A_4 &= \begin{pmatrix} -1 & 0 & 0 & 12 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -0.2 & 0 \\ 0 & 0 & 10 & -0.1 \end{pmatrix}, \\
A_5 &= \begin{pmatrix} -2 & 0 & 0 & 0 & 12 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -0.2 & 0 \\ 0 & 0 & 0 & 10 & -0.1 \end{pmatrix}.
\end{align*}
\]

Ordinary differential equation systems arising from compartmental modeling are ubiquitous in mathematical modeling (see for example [2] or [3] and references therein). For a viral infection compartmental model, the study of local stability involves the linear system \( \frac{d\vec{x}}{dt} = \vec{A} \vec{x} \) with \( \vec{A} \) as defined above [3]. The entries of \( \vec{A} \) as given above represent transition rates between the compartments represented by the entries in \( \vec{x} \). The values \( a \) and \( c \) represent a feed forward process and give the rates per unit time for exiting compartment \( x_2 \) and entering compartment \( x_1 \), respectively. Similarly, the values \( d \) and \( b \) represent a feedback process and give the rates per unit time for exiting compartment \( x_1 \) and entering compartment \( x_2 \), respectively. In the viral infection compartmental model example, \( x_1 \) and \( x_2 \) represent densities of virons and infected cells, respectively. These models can be easily nested inside models with an 'eclipse' or 'latency' phase [3] and the corresponding linearized model would take the form \( \frac{d\vec{x}_3}{dt} = A_3 \vec{x}_3 \) where \( \vec{x}_3 = \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix} \). The new compartment \( x_3 \) serves to delay the process of data transitioning to compartment \( x_1 \) from the initial compartment. In this feed forward process, \( e_3 \) is the rate of transmission from compartment \( x_3 \) to compartment \( x_2 \). The resultant zero column sum indicates no gain or loss from compartment \( x_3 \). Continuing to add latency phases with feedback from the last compartment to the first results in the sequence of matrices defined above. Note that the solutions of the system \( \frac{d\vec{x}}{dt} = A_n \vec{x}_n \) are given by \( \vec{x}_n(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \vec{v}_i \), where \( \lambda_i \) and \( \vec{v}_i \) are the eigenvalues and eigenvectors of \( A_n \) and \( c_i \) are constants chosen to match the initial conditions of the system.

### 3 Preliminary results

Since \( A_{n-1} \) is the Schur complement of \( (A_n)_{1,1} = -e_n \) in \( A_n \), it follows from Definition 1 that \( \det(A_n) = -e_n \det(A_{n-1}) \). Hence, the matrices \( A_n \) are all invertible. Moreover, we have

\[
\begin{align*}
\det(A_3) &= (-e_3) \det(A) = -e_3 (ad - bc) = (-1)^4 e_3 (bc - ad) > 0 \\
\det(A_4) &= (-e_4) \det(A_3) = -e_3 e_4 (bc - ad) = (-1)^5 e_3 e_4 (bc - ad) < 0 \\
&\quad \vdots \\
\det(A_n) &= (-e_n) \det(A_{n-1}) = (-1)^{n+1} e_3 e_4 \cdots e_n (bc - ad).
\end{align*}
\]

We have proved the following lemma.
Lemma 1. The matrices $A_n$ are all invertible and we can write

$$\det(A_n) = (-1)^{n+1} e_3 \cdots e_n (bc - ad) = (-1)^n \left( \prod_{k=3}^{n} e_k \right) \det(A).$$

(1)

Lemma 2. For each $n \geq 3$, $A_n$ has a unique positive eigenvalue of multiplicity one.

Proof: We prove the lemma by deriving a general formula for the characteristic polynomial of $A_n$. First, note that the characteristic polynomials $\det(x I - A_3)$ and $\det(x I - A_4)$ are given by

$$\det(x I - A_3) = (x + e_3)(x + a)(x + d) - bce_5,$$
$$\det(x I - A_4) = (x + e_4)(x + e_3)(x + a)(x + d) - bce_5 e_4.$$ 

Expanding these expressions, we obtain

$$\det(x I - A_3) = x^3 + \sigma_1(a, d, e_3)x^2 + \sigma_2(a, d, e_3)x + (-1)^3 \det(A_3),$$
$$\det(x I - A_4) = x^4 + \sigma_1(a, d, e_3, e_4)x^3 + \sigma_2(a, d, e_3, e_4)x^2 + \sigma_3(a, d, e_3, e_4)x + (-1)^4 \det(A_4).$$

where

$$\sigma_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$$

is the $k$th elementary symmetric polynomial in the alphabet $x_1, \ldots, x_n$. Continuing this way, we see that the characteristic polynomial $f_n(x) = \det(x I - A_n)$ of $A_n$ can be written as

$$f_n(x) = x^n + \sum_{i=1}^{n-1} \sigma_i(a, d, e_3, \ldots, e_n)x^{n-i} + (-1)^n \det(A_n)$$

for all $n \geq 3$. Since $a, b, c, d$ and $e_n$ for $n \geq 3$ are all positive, $f_n(x)$ has all but one positive real coefficients: the only negative coefficient is $(-1)^n \det(A_n)$ by Lemma 1. This implies that (for instance, by Descartes’ rule of signs) $f_n(x)$ has exactly one positive root.

We also observe that the unique positive eigenvalue of $A_4$ has multiplicity one. Indeed, it is evident from (2) that the first derivative $f_n'(x)$ is positive on $[0, \infty)$. In particular, $f_n'(x)$ cannot have a positive root. \qed

Remark 1 Let $f_n(x) = \det(x I - A_n)$ be the characteristic polynomial of $A_n$. As we have seen in the proof of Lemma 2, we have $f_n(x) > 0$ for $x \in [0, \infty)$. In particular, $f_n(x)$ is increasing over $[0, \infty)$.

4 Eigenvalues, main theorem and stability

We will denote the unique positive eigenvalue of $A_n$ by $\lambda_n$ for $n \geq 3$. We show that the sequence \{ $\lambda_n$ \} is a decreasing convergent sequence.

Lemma 3. The characteristic polynomial $f_n(x)$ of $A_n$ admits the following expression:

$$f_n(x) = (x + e_n)f_{n-1}(x) + \sigma_{n-1}(b, c, e_3, \ldots, e_{n-1})x.$$ 

(3)

Proof: Recall that the characteristic polynomial of $A_n$ is given by

$$f_n(x) = x^n + \sum_{i=1}^{n-1} \sigma_i(a, d, e_3, \ldots, e_n)x^{n-i} + (-1)^n \det(A_n).$$

We can write

$$\sigma_i(a, d, e_3, \ldots, e_n) = \sigma_i(a, d, e_3, \ldots, e_{n-1}) + e_n \sigma_{i-1}(a, d, e_3, \ldots, e_{n-1})$$
Thus, multiplying both sides of the above equation by $2\text{.}$

Now, for simplicity in the notation we write $\sigma_i = \sigma_i(a, d, e_3, \ldots, e_n)$ and $\sigma_0 = 1.$ Using the observations above we have

$$f_n(x) = x^n + \sum_{i=1}^{n-1} \sigma_i x^{n-i} + e_n \sum_{i=1}^{n-1} \sigma_{i-1} x^{n-i} + (-1)^n \det(A_n)$$

$$= x^n + \sum_{i=1}^{n-1} \sigma_i x^{n-i} + e_n \sum_{i=1}^{n-1} \sigma_{i-1} x^{n-i} + (-1)^{n-1} e_n \det(A_{n-1})$$

$$= x^n + \sum_{i=1}^{n-1} \sigma_i x^{n-i} + e_n \left( \sum_{i=1}^{n-1} \sigma_{i-1} x^{n-i} + (-1)^{n-1} \det(A_{n-1}) \right)$$

$$= x^n + \sum_{i=1}^{n-1} \sigma_i x^{n-i} + e_n \left( \sum_{i=0}^{n-2} \sigma_i x^{n-i} + (-1)^{n-1} \det(A_{n-1}) \right)$$

$$= x^n + \sum_{i=1}^{n-1} \sigma_i x^{n-i} + e_n f_{n-1}(x).$$

By (2),

$$f_{n-1}(x) = x^{n-1} + \sum_{i=1}^{n-2} \sigma_i x^{n-i-1} + (-1)^{n-1} \det(A_{n-1}).$$

Multiplying both sides of the above equation by $x$, we obtain

$$xf_{n-1}(x) = x^n + \sum_{i=1}^{n-2} \sigma_i x^{n-i} + (-1)^{n-1} \det(A_{n-1}) x.$$ 

Thus,

$$f_n(x) = x^n + \sum_{i=1}^{n-2} \sigma_i x^{n-i} + \sigma_{n-1} x + e_n f_{n-1}(x)$$

$$= xf_{n-1}(x) - (-1)^{n-1} \det(A_{n-1}) x + \sigma_{n-1} x + e_n f_{n-1}(x)$$

$$= (x + e_n)f_{n-1}(x) + (-1)^{n-1} \det(A_{n-1}) + \sigma_{n-1} x.$$ 

Since $(-1)^n \det(A_{n-1}) = (-1)^n (-1)^n (\sigma_{n-1}(b, c, \ldots, e_{n-1}) - \sigma_{n-1})$, we finally obtain $f_n(x) = (x + e_n)f_{n-1}(x) + \sigma_{n-1}(b, c, e_3, \ldots, e_n) x.$

\[\square\]

**Lemma 4.** Let $\tilde{\lambda}_n$ be any eigenvalue of $A_n$ such that $\tilde{\lambda}_n \neq \lambda_n.$ Then $|\tilde{\lambda}_n| \geq \lambda_n.$

**Proof.** Let's assume $|\tilde{\lambda}_n| < \lambda_n.$ Since we know from Remark 1 that $f_n(x)$ is increasing on $[0, \infty)$, we must have $f_n(|\tilde{\lambda}_n|) < f_n(\lambda_n) = 0.$ However, using (2) and setting $\alpha_i = \sigma_i(a, d, e_1, \ldots, e_{n-1}, e_n)$ we obtain

$$f_n(|\tilde{\lambda}_n|) = |\tilde{\lambda}_n|^n + \sum_{i=1}^{n-1} \alpha_i |\tilde{\lambda}_n|^{n-i} + (-1)^n \det(A_n)$$

$$\geq |(\tilde{\lambda}_n)|^n + \sum_{i=1}^{n-1} \alpha_i |\tilde{\lambda}_n|^{n-i} + (-1)^n \det(A_n)$$

$$\geq |f_n(\tilde{\lambda}_n) - (-1)^n \det(A_n)| + (-1)^n \det(A_n)$$

$$= |(-1)^n \det(A_n)| + (-1)^n \det(A_n) = 0,$$

because $(-1)^n \det(A_n) < 0$ for all $n$ by Lemma 1. Therefore, we must have $|\tilde{\lambda}_n| \geq \lambda_n.$

\[\square\]
Theorem 1. The sequence \( \{\lambda_n\}_{n=3}^\infty \), with \( \lambda_n \) being the unique positive eigenvalue of \( A_n \), is a decreasing convergent sequence.

Proof. We know from Remark 1 that \( f_n'(x) > 0 \) on \([0, \infty)\). Let's assume that \( \lambda_{n-1} < \lambda_n \) for some \( n \). By the mean value theorem, there exists a point \( p_n \in (\lambda_{n-1}, \lambda_n) \) such that

\[
f_n'(p_n) = \frac{f_n(\lambda_n) - f_n(\lambda_{n-1})}{\lambda_n - \lambda_{n-1}} = \frac{-f_n(\lambda_{n-1})}{\lambda_n - \lambda_{n-1}},
\]

where \( f_n(\lambda_n) = 0 \) since \( \lambda_n \) is an eigenvalue of \( A_n \). On the other hand, by (3) we have \( f_n(\lambda_{n-1}) = \sigma_{n-1}(b, c, e_3, \ldots, e_{n-1})\lambda_{n-1} \). Then,

\[
f_n'(p_n) = -\frac{\sigma_{n-1}(b, c, e_3, \ldots, e_{n-1})\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} < 0.
\]

But this is not possible because \( f_n'(x) > 0 \) on \((0, \infty)\). Therefore, \( \{\lambda_n\} \) forms a decreasing sequence. Since the sequence \( \{\lambda_n\} \) is decreasing and bounded from below by 0, it is a convergent sequence.

Proposition 1. Suppose that \( \lambda_n \) converges to \( L \geq 0 \) and the product \( \prod_{k=3}^n e_k \) converges to 0. Then, \( L < 1 \) and there exists \( N \) such that \( \lambda_n < 1 \) for all \( n \geq N \).

Proof. If the product \( \prod_{k=3}^n e_k \) converges to 0, then Lemma 1 indicates that

\[
|\det(A_n)| = |\det(A)| \prod_{k=3}^n e_k \to 0.
\]

We also have

\[
|\det(A_n)| = \lambda_n \prod_{j} |\tilde{\lambda}_{n,j}|,
\]

where the \( \tilde{\lambda}_{n,j} \) are non-positive eigenvalues of \( A_n \). From Lemma 4, we know that each \( \tilde{\lambda}_{n,j} \) satisfies \( |\tilde{\lambda}_{n,j}| \geq \lambda_n \). Moreover, since \( \{\lambda_n\} \) is a decreasing sequence, \( \lambda_n \geq L \). Based on these observations we have

\[
|\det(A_n)| = \lambda_n \prod_{j} |\tilde{\lambda}_{n,j}| \geq L^n.
\]

Therefore, since \( |\det(A_n)| \to 0 \), we must have \( L^n \to 0 \). This is the case if \( L < 1 \) as required.

It is not possible to have \( \lambda_n \geq 1 \) for all \( n \). Otherwise, 1 would be a lower bound for the set \( \{\lambda_n| \ n \geq 3\} \). But this contradicts the fact that \( L < 1 \). Therefore, there exists some \( N \) such that \( \lambda_N < 1 \). Because \( \{\lambda_n\} \) is a decreasing sequence, we must have \( \lambda_n < 1 \) for all \( n \geq N \) as claimed.

Proposition 2. The eigenvector of \( A_n \) corresponding to \( \lambda_n \) consists of all non-zero entries with the same sign.

Proof. This is a consequence of the structure of \( A_n \). When solving for the eigenvector, each component \( v_k \) of the eigenvector is a positive multiple of the prior component for \( k = 2, 3, \ldots, n \) and \( v_1 \) is a positive multiple of \( v_n \). If any component of the eigenvector were to be zero, then the vector would be the zero vector and thus it would fail to be an eigenvector.

Recall that \( \lambda_n \) is the unique positive eigenvalue of \( A_n \) with multiplicity 1. Therefore, \( f_n(x) \) is divisible by \((x - \lambda_n)\) and the quotient \( g_n(x) = \frac{f_n(x)}{x - \lambda_n} \) is a degree \( n - 1 \) monic polynomial which does not vanish at \( \lambda_n \). All the remaining \( n - 1 \) eigenvalues of \( A_n \) are the roots of \( g_n(x) \). In the following theorem, we determine the location and structure of the roots of the characteristic polynomials \( f_n(x) \) for small values of \( n \). One can consult [4], [5], [6], [7], [8] for more information on various techniques to locate and characterize the roots of a polynomial.

Theorem 2. All the eigenvalues of \( A_n \) except \( \lambda_n \) have negative real parts for \( n = 3 \) and \( n = 4 \).
Proof. The characteristic polynomials of $A_3$ and $A_4$ are given by
\begin{align*}
f_3(x) &= x^3 + \sigma_1(a, d, e_3)x^2 + \sigma_2(a, d, e_3)x - \det(A_3) \\
f_4(x) &= x^4 + \sigma_1(a, d, e_3, e_4)x^3 + \sigma_2(a, d, e_3, e_4)x^2 + \sigma_3(a, d, e_3, e_4)x + \det(A_4).
\end{align*}
By applying polynomial division we obtain
\begin{align*}
g_3(x) &= x^2 + (\sigma_1(a, d, e_3) + \lambda_1)x + (\sigma_2(a, d, e_3) + \sigma_1(a, d, e_3)\lambda_3 + \lambda_2^3) \\
g_4(x) &= x^3 + a_1x^2 + a_2x + a_3,
\end{align*}
where
\begin{align*}
a_1 &= \sigma_1(a, d, e_3, e_4) + \lambda_4 \\
a_2 &= \sigma_2(a, d, e_3, e_4) + \sigma_1(a, d, e_3)\lambda_4 + \lambda_4^2 \\
a_3 &= \sigma_3(a, d, e_3, e_4) + \sigma_2(a, d, e_3, e_4)\lambda_4 + \sigma_1(a, d, e_3, e_4)\lambda_4^2 + \lambda_4^3.
\end{align*}
Note that all the coefficients of $g_3(x)$ and $g_4(x)$ are positive. To show that all roots of $g_3(x)$ and $g_4(x)$ have negative real parts, we will apply Routh-Hurwitz criterion [2].

Case $n = 3$
In this case it is sufficient to note $\sigma_1(a, d, e_3) + \lambda_3 > 0$ and $(\sigma_2(a, d, e_3) + \sigma_1(a, d, e_3)\lambda_3 + \lambda_2^3) > 0$.

Case $n = 4$
This case is less trivial. By Routh-Hurwitz criterion it is sufficient to show that $a_1 > 0$, $a_3 > 0$ and $a_2 - a_3 > 0$. The positivity of $a_1$ and $a_3$ is obvious. Now, for simplicity in the notation we will write $\sigma_i = \sigma_i(a, d, e_3, e_4)$ and $\lambda_i = \lambda$. Then,
\begin{align*}
a_1a_2 - a_3 &= (\sigma_1 + \lambda)(\sigma_2 + \sigma_1\lambda + \lambda^2) - (\sigma_3 + \sigma_2\lambda + \sigma_1\lambda^2 + \lambda^3) \\
&= \sigma_1\sigma_2 - \sigma_3 + \sigma_1\lambda^2 + \sigma_1^2\lambda.
\end{align*}
By definition, $\sigma_1 = a + d + e_3 + e_4$, $\sigma_2 = ad + ae_3 + ae_4 + de_3 + de_4 + e_3e_4$ and $\sigma_3 = ade_3 + ade_4 + ae_3e_4 + de_3e_4$. Then, we have
$$\sigma_1\sigma_2 = ade_3 + ade_4 + ae_3e_4 + de_3e_4 + \Theta = \sigma_1 + \Theta,$$
where $\Theta$ is positive. Therefore, $\sigma_1\sigma_2 - \sigma_3 = \Theta$ and so $a_1a_2 - a_3 > 0$, as required. \hfill \Box

Note that the number of roots of $g_3(x)$ and $g_4(x)$ having negative real part does not depend on the choice of the initial matrix $A$ or the choice of the sequence $(e_n)$. It is curious to ask if the same can be said for $n > 4$. Unfortunately, this is not true for arbitrary $A$. For instance, in Example 1 we have
$$A_5 = \begin{pmatrix}
-2 & 0 & 0 & 0 & 12 \\
2 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & -0.2 & 0 \\
0 & 0 & 0 & 10 & -0.1
\end{pmatrix},$$
and $0.0927315107187 \pm 2.77510340056i$ are two complex eigenvalues of $A_5$ with positive real parts.

5 Discussion and further work

For a compartmental model as described in the Section 2, an interesting question is if a memory of non-zero initial data for the first compartment will appear in the last compartment as an increasing number of latency phases are added to the system. The results here, specifically the fact that the eigenvector corresponding to the unique positive real eigenvalue has all non-zero components, indicate that, regardless of number of phases, this memory will appear in the last compartment.
Although it is not true that all the eigenvalues (except the unique positive one) have negative real parts for all \(n\), it is interesting to ask if it is possible to acquire this situation by imposing some restrictions on either the matrix \(A\) or the sequence \(\{e_n\}\). Based on several simulations we have the following conjecture.

**Conjecture** If the product \(\prod_{k=3}^n e_k\) is sufficiently small and the positive eigenvalue of the matrix \(A\) is less than 1, then all the eigenvalues of \(A_n\) (except the unique positive one) have negative real part for any \(n\).

Indeed, the determinants that arise from application of the Routh-Hurwitz criterion involve several terms, some of which are negative. When rewriting these determinants as the sum of positive and negative parts, the negative parts can be written as a function of the unique positive eigenvalue \(\lambda_n\), and the numbers \(\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\) defined in the proof of Lemma 2. Our computations using a computer algebra system indicate that, under the conditions of the conjecture, the positive part of the determinant will be greater than the absolute value of the negative part making all such determinants positive as required by the Routh-Hurwitz criterion.

**References**