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The Physics of Moving Bodies

Diego Castano
Nova Southeastern University, castanod@nova.edu

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Diego J. Castaño[a)]
Nova Southeastern University, 3301 College Ave., Fort Lauderdale, Florida 33314, USA

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Abstract: An alternative theoretical development of Special Relativity is presented in which Electrodynamics is not invoked. In deriving the Lorentz transformation in this manner, the existence of a maximal speed for all physical phenomena stands out. The mass-energy relation is also derived without reference to light.
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Résumé: Un développement théorique alternative de la relativité restreinte est présenté dans lequel l'électrodynamique n'est pas invoquée. En dérivant la transformation de Lorentz de cette manière, l'existence d'une vitesse maximale pour tous les phénomènes physiques ressort. La relation masse-énergie est également dérivée sans référence à la lumière.

Key words: Special Relativity; Einstein; Lorentz Transformation; Spacetime; Symmetry; Causality; Metric Tensor; Mass-Energy.

I. INTRODUCTION

Special Relativity (SR) was developed by Albert Einstein and relied ostensibly on Electrodynamics, a theory not fully developed until 1873.[1] Yet there is nothing in SR's theoretical development that requires post-Newtonian physics. Consequently, Newton, or an 18th century successor, could have in principle derived SR. Electrodynamics "only" provides a possible context for empirical determination of, what Reichenbach refers to as, the first-signal.[2] Indeed, extricating Electrodynamics from SR emphasizes the critical fact that there exists a universal limit to speed in classical physics regardless of phenomenal context. This point is well worth emphasizing to students of SR. The view taken here is that it is more remarkable that there is a limiting speed inherent in physics than that light happens to be a phenomenon that expresses it. Of course, that this speed is finite (obtained first using light) is what truly distinguishes the classical physics world view from the post-Einstein one.

In this article, an alternative (unusual but instructive) and relatively simple derivation of SR is presented in the simplified setting of $1+1$ dimensions. Omitting any reference to Electrodynamics, the Lorentz transformation, and consequently the basic kinematic results of SR, is derived. We are aware of at least one previous and different approach to make light vanish from the Lorentz transformation derivation.[3] The present, independent approach is pursued further, and by invoking some modern mathematical developments, it is shown that the Minkowski metric tensor and the mass-energy relation can also be derived.

II. SPACE AND TIME MIXING

To begin we assume a system of coordinates, S, "in which the equations of Newtonian mechanics hold good."[1] Since the laws of physics are empirically translationally and rotationally invariant, we expect that any translational/rotational (T/R) transformation of the original system will not invalidate the laws. We define an equivalence relation among systems of coordinates based on T/R and call the equivalence classes inertial reference frames (IRFs). We seek spacetime transformations (i.e., non-T/R) between IRFs (which are referred to as boosts) that respect Newtonian mechanics and follow from intuitively consistent constraints. For the sake of simplicity, we work in one space dimension and, of course, time, so $1+1$ dimensions. The points in this (Affine) space are known as events, $E(t,x)$.

Constraint 1: We constrain a general boost by ensuring that it respect the laws of motion. The first law of motion implies that force free bodies exhibit straight worldlines (see Fig. 1). The transformation we seek must transform straight worldlines into straight worldlines, therefore the transformation must be linear

$$t' = At + Bx, \tag{1}$$

$$x' = Ct + Dx. \tag{2}$$

A more detailed discussion of the validity of this form can be found in Ref. 4. In matrix form

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \mathbf{L}\begin{pmatrix} t \\ x \end{pmatrix}, \tag{3}$$

where

$$\mathbf{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{4}$$

We can consider how velocity transforms

$$u' = \lim_{\Delta t' \to 0} \frac{\Delta x'}{\Delta t'} = \frac{C + Du}{A + Bu}. \tag{5}$$

[a)]castanod@nova.edu

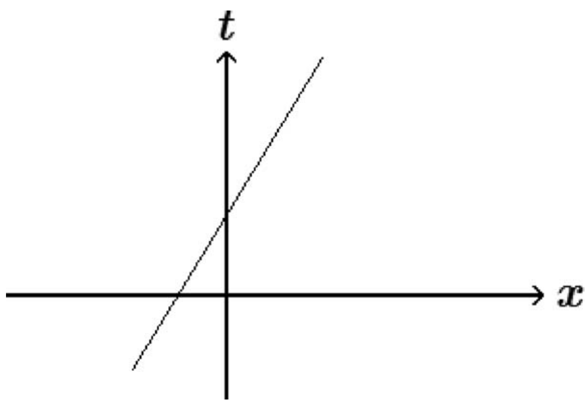

FIG. 1. A net-force-free worldline.

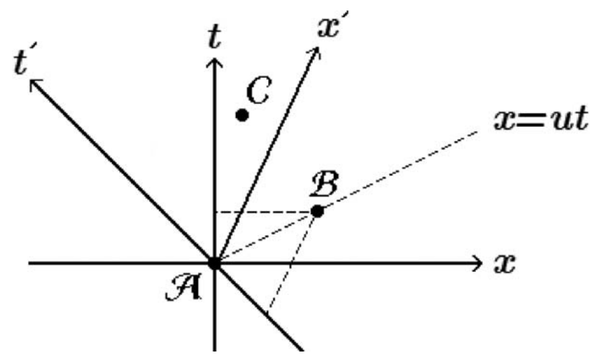

FIG. 2. Example of acausality in the $\sigma = 1$ case.

This form implies that the general transformation results in a new system, S', in motion relative to S and with relative speed $|C/D|$. To see this, consider a stationary body with respect to S' (i.e., with $u' = 0$). We identify the relative boost velocity as

$$u = -\frac{C}{D} \equiv v. \tag{6}$$

Constraint 2: We assume spatial symmetry with respect to relative motion (i.e., so-called standard signal synchrony[5]) so that a body at rest in S ($u = 0$) has equal and opposite velocity in S'

$$u' = \frac{C}{A} = -v. \tag{7}$$

Eliminating C and D yields

$$\mathbf{L}(v) = A(v)\begin{pmatrix} 1 & b(v) \\ -v & 1 \end{pmatrix}. \tag{8}$$

Implicit in this spatial symmetry constraint is invariance of the transformation under $x \to -x$, $v \to -v$, so that

$$b(-v) = -b(v), \tag{9}$$

$$A(-v) = A(v). \tag{10}$$

Constraint 3: We assume that given $\mathbf{L}(v)$ the inverse exists and that $\mathbf{L}^{-1}(v) = \mathbf{L}(-v)$ so that

$$\mathbf{L}(v)\mathbf{L}(-v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{11}$$

This implies

$$A(v) = \frac{\alpha}{\sqrt{1 + bv}}, \tag{12}$$

where $\alpha = \pm 1$. We will return to this parameter in Constraint No. 5 below.

Constraint 4: We assume that boosting is transitive. Given two consecutive boosts with associated velocities, v_1 and v_2, there is an equivalent single boost for some velocity v_{12}

$$\mathbf{L}(v_1)\mathbf{L}(v_2) = \mathbf{L}(v_{12}). \tag{13}$$

This implies $b = kv$, where k is a constant with units of inverse speed squared. Let $k = \sigma/c^2$, where c has units of speed and $\sigma = \pm 1$. We consider this parameter below.

The second and third laws of motion (conservation of momentum) were not invoked because they turn out to be redundant given Constraint Nos. 3 and 4. In summary, the general transformation between IRFs we seek has the following form:

$$\mathbf{L}(v) = \frac{\alpha}{\sqrt{1 + \sigma(v/c)^2}}\begin{pmatrix} 1 & \sigma v/c^2 \\ -v & 1 \end{pmatrix}. \tag{14}$$

Constraint 5: We assume that temporal order between causally connected events (e.g., two events on the trajectory of a moving body of speed u) is unchanged by a boost

$$\frac{\Delta t'}{\Delta t} = \frac{\alpha}{\sqrt{1 + \sigma(v/c)^2}}\left(1 + \sigma\frac{uv}{c^2}\right) > 0. \tag{15}$$

If $\alpha = -1$, it is evident that for $u = 0$ this is violated regardless of σ, so $\alpha = 1$.

We now consider the parameter σ and first take it to be $\sigma = +1$, therefore

$$\mathbf{L}(v) = \frac{1}{\sqrt{1 + (v/c)^2}}\begin{pmatrix} 1 & v/c^2 \\ -v & 1 \end{pmatrix} \tag{16}$$

with corresponding velocity composition

$$u' = \frac{u - v}{1 + uv/c^2}. \tag{17}$$

We note that, for example (see Fig. 2), if $u > 0$ and $v < 0$ with $|uv| > c^2$ (hence $-v/c^2 > 1/u$), the causal temporal order condition (15) is violated. In the figure, the $x'-$ axis is the line $t = (-v/c^2)x$; the $t'-$ axis is the line $t = (1/v)x$. Note that these two axes are not perpendicular. For any event $\mathcal{B}$ temporally following event $\mathcal{A}$ on the causal worldline $x = ut$ in system S, we find that event $\mathcal{B}$ occurs before $\mathcal{A}$ in system S'.

To attempt to remedy this problem, we constrain all speeds, i.e., interpret c as some universal speed limit

$$|u|,\ |v| < c. \tag{18}$$

This condition however fails to be upheld by velocity composition. For example, consider the case $c > u > c/2$ and $c > -v > c/2$, then

$$u' = \frac{u + (-v)}{1 - u(-v)/c^2} > c. \tag{19}$$

We are therefore left with one last possible transformation with $\sigma = -1$

$$\mathbf{L} = \frac{1}{\sqrt{1 - (v/c)^2}} \begin{pmatrix} 1 & -v/c^2 \\ -v & 1 \end{pmatrix}, \tag{20}$$

$$u' = \frac{u - v}{1 - uv/c^2}. \tag{21}$$

If we demand that the radicand be positive, then there is a limiting boost speed $|v| < c$. Since the speed of a body is independent of the boost speed, the causal temporal order condition (15) requires all speeds $|u| < c$.

Velocity composition does not belie this case. Let $0 < (-v) = \beta_v c < c$, $u = \beta c$, $u' = \beta' c$. Since $0 < \beta_v < 1$, and $0 < 1 - \beta < 1$, it follows that:

$$\begin{aligned} \beta_v(1 - \beta) &< 1 - \beta \\ \beta + \beta_v &< 1 + \beta\beta_v \\ \frac{\beta + \beta_v}{1 + \beta\beta_v} &< 1 \end{aligned} \tag{22}$$

therefore

$$u' = \frac{u - v}{1 - uv/c^2} < c. \tag{23}$$

Note that even for c finite, every observer measures the phenomenon associated with this speed to have the same speed

$$c' = \frac{c - v}{1 - v/c} = c. \tag{24}$$

In other words, whatever moves, or propagates, at the limiting speed does not suffer from the relativity of motion. The converse is also true. Einstein's second postulate identifies electromagnetic waves as having this characteristic.

So, we have found a consistent spacetime transformation upholding the laws of motion and satisfying various straightforward physical/mathematical constraints. Even without exploring the notable kinematic consequences, such as time dilation and length contraction, the conclusion that there is a speed limit in physics (exposed by the causal structure imposed by Constraint No. 5) is already remarkable. The value of this speed limit, or of the one undetermined constant, c, is now an empirical matter. Before we consider experiment, we display the form of the ($x-$ directed) Lorentz transformation of the spacetime point $E(t, x, y, z)$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right), \tag{25}$$

$$x' = \gamma(x - vt), \tag{26}$$

$$y' = y, \tag{27}$$

$$z' = z, \tag{28}$$

with

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c}. \tag{29}$$

We have assumed the primed and unprimed spatial coordinate systems have parallel axes and that the relative motion is along the $x-$ axis, so the two spatial coordinates perpendicular to x are constant. The resulting transformation of the velocity components is

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2}, \tag{30}$$

$$u'_y = \frac{1}{\gamma}\frac{u_y}{1 - u_x v/c^2}, \tag{31}$$

$$u'_z = \frac{1}{\gamma}\frac{u_z}{1 - u_x v/c^2}. \tag{32}$$

III. EXPERIMENT

Had the history of physics followed the route presented above, the search for the value of c could "simply" come down to comparing relative velocities of any body in an experiment with sufficient precision. Alternatively, and as already pointed out, if a phenomenon is identified whose speed is independent of IRF, then this speed must be c. We now consider another characteristic of such maximum speed phenomena that has actual historical coincidence. Suppose a phenomenon known to propagate isotropically at speed u_* in one IRF, say, S, is subjected to experimentation in another IRF, S', in relative motion, v, with respect to S. Using reflectors (devices that reverse the direction of movement), the round-trip time from start to reflector and back is measured in two perpendicular directions, one of which is the direction of motion. Taking the two paths to be equally long in S' (denoted by L in the following) and employing Eqs. (30), and (31) or (32), the theoretical round-trip times in S' are

$$\Delta t_{\|} = \frac{2L}{\gamma^2}\frac{1}{u_*(1 - v^2/{u_*}^2)}, \tag{33}$$

$$\Delta t_{\perp} = \frac{2L}{\gamma}\frac{1}{u_*(1 - v^2/{u_*}^2)^{1/2}}. \tag{34}$$

If these times are found to be equal regardless of the boost speed, v, then the propagation speed of the phenomenon, u_*, equals c, the universal speed limit. This is, of course, what Michelson and Morley inadvertently discovered of light propagation.

IV. THE METRIC TENSOR

In the following, we employ some modern mathematical methods (not available to Newton) to further probe the alternative history we are entertaining. In the purely spatial context, the dot product ensures invariance under rotations. Indeed, vectors (or 3-vectors to make clear the dimensional setting) are defined in terms of their transformation properties under rotational transformations in the Affine space. We now use the Lorentz transformation derived above to define 4-vectors by using the transformation rule for the spacetime point $E(t,x,y,z)$, Eqs. (25)–(28), as a template. We first introduce some conventions that will result in more elegant notation in what follows. Time will be scaled by c and measured in $c\cdot$ seconds, for example. For ease of indexing, the time parameter will be denoted x^0 $(= ct)$, and the space parameters will be denoted x^1 $(= x)$, x^2 $(= y)$, and x^3 $(= z)$. These will now be considered components of a difference vector in the associated vector space. Under the ($x-$directed) Lorentz transformation, a 4-vector, $\bar{v} = \sum v^{\mu}\hat{e}_{\mu} = \left(v^0, v^1, v^2, v^3\right)$, transforms as follows:

$$\mathbf{V}' = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{V} = \Lambda_x \mathbf{V}, \qquad (35)$$

where $\mathbf{V}$ is the column matrix of components. The general form of an inner product is

$$\bar{v} \circ \bar{w} \equiv \sum\sum v^{\mu}\eta_{\mu\nu}w^{\nu} = \mathbf{V}^T\eta\mathbf{W}, \qquad (36)$$

where η is a 4×4 nonsingular matrix that ensures symmetry, bilinearity, and nondegeneracy of the product. We wish that this product remain invariant under a Lorentz transformation

$$\mathbf{V}'^T\eta\mathbf{W}' = \mathbf{V}^T\eta\mathbf{W}, \qquad (37)$$

which constrains the form of the metric tensor, η,

$$\Lambda_x{}^T\eta\Lambda_x = \eta. \qquad (38)$$

It is sufficient to consider the 2×2 (tx) submatrix and show that

$$\eta_{2\times 2} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (39)$$

with undetermined a. This is, of course, the Minkowski metric tensor. The difference vector between two simultaneous (in a given IRF) spacetime events, $\Delta\bar{s} = (0, \Delta\vec{r})$, would have squared norm

$$\Delta\bar{s}^2 = a\left(-\Delta r^2\right). \qquad (40)$$

This would coincide with the Euclidean norm if $a = -1$, but we will adopt the particle physics convention and take $a = 1$.

V. MOMENTUM, ENERGY, AND MASS

The differential difference vector between two spacetime events on the worldline of a constant velocity body, $d\bar{s}^2 = (cdt, udt, 0, 0)$, has squared norm

$$d\bar{s}^2 = c^2dt^2\left(1 - \beta_u{}^2\right). \qquad (41)$$

In a comoving IRF with $v = u$, $u' = 0$ and $d\bar{s}^2 = c^2dt'^2$, and we thus recognize this as the time measured by a clock carried along with the body. This motivates the definition of the proper time interval

$$\begin{aligned} d\tau &= \frac{1}{c}\sqrt{d\bar{s}^2} \\ &= \frac{dt}{\gamma_u}, \quad \gamma_u = \frac{1}{\sqrt{1-\beta_u{}^2}} \end{aligned} \qquad (42)$$

with the second row equality following from the invariance property of $d\bar{s}^2$. We can now use this scalar quantity, $d\tau$, and a spacetime event difference 4-vector, $d\bar{s} = (cdt, \vec{u}dt)$, to construct a new 4-vector

$$\frac{d\bar{s}}{d\tau} = \gamma_u(c, \vec{u}). \qquad (43)$$

With this 4-velocity, the obvious guess for a 4-momentum is

$$\bar{p} = m\frac{d\bar{s}}{d\tau} = (mc\gamma_u, \vec{p}), \qquad (44)$$

$$\vec{p} = \gamma_u m\vec{u}, \qquad (45)$$

with $\bar{p}^2 = (mc)^2$. It is clear from its construction that if conserved in one IRF, then this quantity is conserved in all others. To show that it is indeed conserved, we would consider a Lorentz invariant action for a moving body ($\propto \int d\tau$) and invoke Noether's theorem.[6] We will instead follow the usual route and assume the validity of Newton's second law (as has been done implicitly with Constraint Nos. 3 and 4) with the new definition of momentum

$$\vec{F} = \frac{d\vec{p}}{dt} = \gamma_u^3 m\vec{a} \qquad (46)$$

assuming $\vec{u}$ and $\vec{a}$ are parallel. As is standard practice, we then use the work-energy theorem to derive the kinetic energy (KE) formula

$$\begin{aligned} KE = W &= \int Fdx' = \int \frac{dp'}{dt}dx' = \int_0^p u'dp' \\ &= mc^2(\gamma_u - 1) \end{aligned} \qquad (47)$$

The 4-momentum can also be used to give some insight into the maximal speed. We can use definition (45) to express speed in terms of mass and momentum

$$u = \frac{pc}{\sqrt{p^2 + m^2c^2}}, \quad p = |\vec{p}| \tag{48}$$

from which it follows that masslessness is a sufficient condition for maximal speed. It is not difficult to show that it is also a necessary condition. Mathematically, at least, the possibility of imaginary mass particles, known as tachyons, would imply that the limiting speed, c, is instead a deliminating speed; tachyons would never travel slower than c. There are even good physical arguments for their classical viability.[7] Nevertheless, tachyonic phenomena violate Constraint No. 5 above which forces us to rule them out as realizable.

We stress that all of these theoretical developments do not depend on the value of c nor have any reliance on Electromagnetism. We now derive the mass-energy relation in this same way and depart from the typical invocation of Electromagnetism to do so. We consider the collision of two mass-identical bodies moving with equal and opposite velocities before the collision and whose outcome is a new stationary body

$$\bar{p}_+ = \gamma_u m(c, u, 0, 0), \tag{49}$$

$$\bar{p}_- = \gamma_u m(c, -u, 0, 0), \tag{50}$$

$$\bar{p}_0 = M(c, 0, 0, 0). \tag{51}$$

Assuming momentum is conserved, $\bar{p}_+ + \bar{p}_- = \bar{p}_0$, and therefore $M = 2m\gamma_u > 2m$. Since before the collision the moving masses had KE, and after the collision, the stationary one does not, if we assume energy is conserved as well as momentum, we are forced to conclude that there is energy in mass

$$E = E_0(m) + KE \tag{52}$$

and the mass energy must be linear (i.e., $E_0(m_1 + m_2) = E_0(m_1) + E_0(m_2)$, with $E_0(m) = \xi m$ and ξ a constant); otherwise the mere aggregation of bodies would violate energy conservation. Our example collision then implies

$$\begin{aligned} E_0(M) &= 2KE + E_0(2m) \\ E_0(M - 2m) &= 2KE \\ \xi(2m\gamma_u - 2m) &= 2mc^2(\gamma_u - 1) \\ \xi &= c^2 \end{aligned} \tag{53}$$

so that

$$E = mc^2 + KE, \tag{54}$$

$$E = \gamma_u mc^2, \tag{55}$$

$$E = \sqrt{(pc)^2 + (mc^2)^2}. \tag{56}$$

VI. CONCLUSIONS

Most textbooks present the development of the theory of SR in similar fashion by following Einstein's own development and considering the implications of his two postulates. In the present article, a different theoretical development that does not invoke light has been presented. Within a classical context, using Newtonian laws of motion and intuitive physics and mathematical constraints, the Lorentz transformation and hence the Special Theory of Relativity was derived in the simplified setting of $1 + 1$ dimensions. By eliminating any reference to Electrodynamics however, emphasis was placed on the consequential result that there is a limiting speed for all physical phenomena. This limiting speed is a result of the causal structure imposed by Constraint No. 5 in the analysis. Had Newton, or an imminent disciple, derived such a result, this would have called into question any subsequent historical consideration of superluminal speed like Laplace's with regard to gravity. The results of the Michelson–Morley experiment were shown to represent the discovery that light is a maximal speed phenomenon. With the appropriate mathematical tools, Newton could have concluded that the natural setting for spacetime is non-Riemannian or Minkowski (in today's terminology). Finally, the mass-energy relation was derived, again without recourse to Electromagnetism.

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