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Uniform Convergence of Polynomials Associated with Varying Jacobi Weights

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UNIFORM CONVERGENCE OF POLYNOMIALS ASSOCIATED WITH VARYING JACOBI WEIGHTS

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Dedicated to Prof. W.J. Thron on the occasion of his 70th birthday

ABSTRACT. In this paper we determine the functions on $[-1,1]$ that are uniform limits of weighted polynomials of the form $(1-x)^{\alpha_n}(1+x)^{\beta_n}p_n(x)$, where $\deg p_n \le n$, $\lim_{n\to\infty}\alpha_n/n = \theta_1 \ge 0$ and $\lim_{n\to\infty}\beta_n/n = \theta_2 \ge 0$. Estimates for the rate of convergence are also obtained. Our results confirm a conjecture of Saff for $w(x) = (1-x)^{\theta_1}(1+x)^{\theta_2}$, when $\theta_1 > 0$, $\theta_2 > 0$, and extend previous results of G.G. Lorentz and M. v. Golitschek, and Saff and Varga for incomplete polynomials.

1. Introduction. The introduction of "incomplete polynomials" by G.G. Lorentz [**4**] in 1976 has led to an extensive study of polynomials with varying weights. Among the more recent results is the solution of Freud's conjecture [**5**], and strong asymptotics for a family of extremal polynomials associated with exponential weights on **R** [**7**]. The essential question which serves as the starting point for these investigations is the following:

Suppose $w : \mathbf{R} \to \mathbf{R}$ is a nonnegative weight function continuous on its support Σ. An important problem is the characterization of limit functions of sequences of weighted polynomials of the form

$$[w(x)]^n p_n(x), \quad n = 1, 2, \dots ,$$

where $p_n \in \mathcal{P}_n$, the collection of all algebraic polynomials of degree at most n.

Mhaskar and Saff [**8**] proved that the sup norm of $[w(x)]^n p_n(x)$ over Σ actually "lives" on some (smallest) compact set $\mathcal{S} \subset \{x \in \Sigma : w(x) \neq 0\}$ which is independent of n and p_n. The connection between this fundamental result and our problem is that, in several important cases,

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$f \in C(\Sigma)$ and $\operatorname{supp}(f) \subset \mathcal{S}$ is the necessary and sufficient condition for the existence of a sequence of weighted polynomials $\{[w(x)]^n p_n(x)\}_{n=1}^{\infty}$ that converges uniformly to $f(x)$ on Σ. We refer the reader to [**4,11**] for $w(x) = x^s$, $s > 0$, $\Sigma = [0,1]$; [**9**] for $w(x) = e^{-x}$, $\Sigma = [0,\infty)$; [**6**] for $w(x) = e^{-|x|^{\alpha}}$, $\alpha > 1$, $\Sigma = (-\infty,\infty)$.

The purpose of the present paper is to investigate the corresponding result for the case when $w(x)$ is a Jacobi Weight. We shall give an affirmative answer to Saff, Ullman and Varga's conjecture in [**10**] (cf. part (ii) of Theorem 1). More precisely, we will characterize those functions that are uniform limits on $[-1,1]$ of sequences of the form $\{(1-x)^{\alpha_n}(1+x)^{\beta_n}p_n(x)\}_{n=1}^{\infty}$, where $\lim_{n\to\infty}\alpha_n/n = \theta_1 \geq 0$, $\lim_{n\to\infty}\beta_n/n = \theta_2 \geq 0$ and $p_n \in \mathcal{P}_n$, $n = 1,2,\ldots$.

For the special case when $\{\alpha_n\}_{n=1}^{\infty}$ are integers, $\beta_n = 0$ for all $n = 1,2,\ldots$, the limit functions were found by Saff and Varga in [**11**] and von Golitschek in [**2**] for $\theta_1 > 0$ and by Lorentz in [**4**] for $\theta_1 = 0$. It is not a trivial extension to characterize the limit functions for the general case when $\theta_1 > 0$ and $\theta_2 > 0$. Our method of proof was inspired by a recent work of Lubinsky and Saff [**6**].

The outline of this paper is as follows. In Section 2, we introduce notations, definitions and state our results. Section 3 contains the proofs of the theorems stated in Section 2.

2. Statement of results. Let $I := [-1,1]$. For any set B and function f defined on B, let

$$\|f\|_B := \sup\{|f(x)|; x \in B\}.$$

Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences of nonnegative reals with

$$\lim_{n\to\infty}\frac{\alpha_n}{n} = \theta_1 \quad \text{and} \quad \lim_{n\to\infty}\frac{\beta_n}{n} = \theta_2. \tag{2.1}$$

For $\theta_1 \geq 0$, $\theta_2 \geq 0$, define, as in [**8,10**],

$$\begin{aligned} a &:= a(\theta_1,\theta_2) = \sin(\varphi_1 - \varphi_2), \\ b &:= b(\theta_1,\theta_2) = \sin(\varphi_1 + \varphi_2), \end{aligned} \tag{2.2}$$

where

$$\sin\varphi_1 := \frac{\theta_1+\theta_2}{1+\theta_1+\theta_2}, \qquad \cos\varphi_2 := \frac{\theta_2-\theta_1}{1+\theta_1+\theta_2},$$

and $0 \le \varphi_1 < \pi/2$, $0 < \varphi_2 < \pi$.

One of our main theorems is

Theorem 1. *Let f be a function defined on I and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ be sequences of nonnegative reals satisfying* (2.1). *Then there is a sequence of weighted polynomials $\{(1-x)^{\alpha_n}(1+x)^{\beta_n}p_n(x)\}_{n=1}^{\infty}$, where $p_n \in \mathcal{P}_n$, such that*

$$\lim_{n\to\infty}(1-x)^{\alpha_n}(1+x)^{\beta_n}p_n(x) = f(x) \tag{2.3}$$

uniformly on I if and only if $f \in C(I)$ and

(i) *when $\alpha_n = 0$ and $\beta_n > 0$ for n large enough, $f(x) = 0$ for $x \in [-1, a]$.*

(ii) *when $\alpha_n > 0$ and $\beta_n > 0$ for n large enough, $f(x) = 0$ for $x \in [-1, a] \cup [b, 1]$.*

(iii) *when $\alpha_n > 0$ for infinitely many n and $\theta_1 = 0$, $\beta_n > 0$ for n large enough, $f(x) = 0$ for $x \in \{1\} \cup [-1, a]$.*

(iv) *when $\alpha_n > 0$, $\beta_n > 0$ for infinitely many n, respectively, and $\theta_1 = \theta_2 = 0$, $f(x) = 0$ for $x = 1$ and -1.*

We remark that the Weierstrass theorem gives us the result when $\alpha_n = \beta_n = 0$ for n large enough, and the symmetry between α_n and β_n allows us to get the corresponding results for all the other possible cases of α_n and β_n in (i) and (iii).

The following theorem concerning the rate of convergence generalizes the result in [**2**].

Theorem 2. *Let $\theta_1 \ge 0$, $\theta_2 \ge 0$. Let a, b be as defined in* (2.2). *Let $[c, d]$ be a subinterval of $[a, b]$ with $a < c < d < b$. Then there exist constants $K > 0$ and $\tau > 0$ (only depending on c, d, θ_1 and θ_2) such that, for any $f \in C[c, d]$, there exists $p_n \in \mathcal{P}_n$ such that*

$$\|f(x) - (1-x)^{n\theta_1}(1+x)^{n\theta_2}p_n(x)\|_{[c,d]} \le K\omega\left(f, \frac{1}{n}\right) + O(e^{-\tau n}) \tag{2.4}$$

for $n = 1, 2, \dots,$ *where* $\omega(f, \cdot)$ *is the modulus of continuity of* f *on* $[c, d]$.

Furthermore, f *is analytic on* $[c, d]$ *if and only if there exist* $p_n \in \mathcal{P}_n$ *and* $\tau_1 > 0$ (*depending on* c, d, θ_1, θ_2 *and* f) *such that*

$$\|f(x) - (1-x)^{n\theta_1}(1+x)^{n\theta_2}p_n(x)\|_{[c,d]} = O(e^{-\tau_1 n}). \tag{2.5}$$

3. Proofs. We need the following lemma in our proofs.

Lemma 3.1. *Let* $\{a_{m,n}\}_{m,n=1}^{\infty}$ *be a doubly infinite sequence. If*

$$\lim_{n\to\infty} a_{m,n} = a_m, \quad m = 1, 2, 3, \dots ,$$

and

$$\lim_{m\to\infty} a_m = a,$$

then there exists a sequence $\{m(n)\}_{n=1}^{\infty}$ *such that*

$$\lim_{n\to\infty} a_{m(n),n} = a.$$

The proof of Lemma 3.1 is elementary and is therefore omitted.

Proof of Theorem 1. We divide the proof into several steps.

Proof of (i) *when* $\theta_2 = 0$. Since $a(0,0) = -1$ the necessity is obvious. For sufficiency, we first assume that f is of the form

$$f(x) = (1+x)\sum_{k=0}^{m} a_k(1+x)^k,$$

for some nonnegative integer m.

When $\beta_n > 1$ for n large enough, consider

(3.1)

$$E_n := \inf_{p_n \in \mathcal{P}_n} \left\{ \left\| (1+x) \sum_{k=0}^{m} a_k (1+x)^k - (1+x)^{\beta_n} p_n(x) \right\|_I \right\}$$

$$= \inf_{p_n \in \mathcal{P}_n} \left\{ \left\| \int_{-1}^{x} \left\{ \sum_{k=0}^{m} (k+1) a_k (1+t)^k - \beta_n (1+t)^{\beta_n - 1} p_n(t) \right.\right.\right.$$

$$\left.\left.\left. - (1+t)^{\beta_n} p_n'(t) \right\} dt \right\|_I \right\}$$

$$\leq \inf_{\substack{p_{n,k} \in \mathcal{P}_n \\ k=0,1,\dots,m}} \left\{ \left\| \sum_{k=0}^{m} \int_{-1}^{x} |(k+1) a_k (1+t)^k - (1+t)^{\beta_n - 1} p_{n,k}(t)| \, dt \right\|_I \right\},$$

since any $\sum_{k=0}^{m} p_{n,k}(t)$ with $p_{n,k} \in P_n$ is equal to $\beta_n p_n(t) + (1+t) p_n'(t)$ for some $p_n \in \mathcal{P}_n$. By the Cauchy-Schwarz inequality, we get

(3.2)

$$E_n \leq \sqrt{2} \sum_{k=0}^{m} \inf_{p_{n,k} \in \mathcal{P}_n} \left\{ \int_{-1}^{1} [(k+1) a_k (1+t)^k - (1+t)^{\beta_n - 1} p_{n,k}(t)]^2 \, dt \right\}^{1/2}$$

$$\leq \sum_{k=0}^{m} (k+1) a_k 2^{k+1} \frac{1}{\sqrt{2k+1}} \cdot \frac{|k - \beta_n + 1|}{n + k + \beta_n};$$

here in the last inequality we used an identity of Muntz (cf. Cheney [**1**, p. 196]).

Since $\beta_n / n \to 0$ as $n \to \infty$, we have

$$\lim_{n \to \infty} \frac{|k - \beta_n + 1|}{n + k + \beta_n} = 0, \quad k = 0, 1, \dots, m,$$

and so

$$\lim_{n \to \infty} E_n = 0.$$

This implies that $(1+x) \sum_{k=0}^{m} a_k (1+x)^k$ is a limit function of a sequence of weighted polynomials of form $(1+x)^{\beta_n} p_n(x)$, $p_n \in \mathcal{P}_n$, where $\beta_n / n \to 0$ as $n \to \infty$. Since, by the Weierstrass theorem, any function $f \in C(I)$ with $f(-1) = 0$ is a limit function of a sequence of polynomials of the form $(1+x) \sum_{k=0}^{m} a_k (1+x)^k$, by Lemma 3.1 we see that (2.3) is true for $\beta_n > 1$ when n is large enough. For the remaining case $0 < \beta_n \leq 1$ for infinitely many n, by the above discussion, there exist $q_n(x) \in \mathcal{P}_{n-1}$, $n = 1, 2, \ldots$, such that

$$\lim_{n\to\infty} (1+x)^{\beta_n+1} q_n(x) = \lim_{n\to\infty} (1+x)^{\beta_n} p_n = f(x)$$

uniformly on I, where $p_n \in \mathcal{P}_n$.

Proof of (i) *when* $\theta_2 > 0$. Again we need only to show the sufficiency. (See, for example, [**2**]). If the β_n's are integers for all n large, then Theorem 2.1 in [**11**] (or Theorem 1 in [**2**]) gives the result. So assume that we have infinitely many noninteger β_n. Note that $f(x)$ can be uniformly approximated on I by functions of form

$$h(x) = \begin{cases} 0, & x \in [-1, a], \\ (x-a)p(x), & x \in [a, 1], \end{cases}$$

where $p(x)$ is a polynomial. And we can write $h(x) = r(x)s(x)$ with

$$r(x) := \begin{cases} 0, & x \in [-1, a], \\ \sqrt{x-a}, & x \in [a, 1], \end{cases}$$

and

$$s(x) := \begin{cases} 0, & x \in [-1, a], \\ \sqrt{x-a}\, p(x), & x \in [a, 1]. \end{cases}$$

Now, from the proof of (i) when $\theta_2 = 0$, we know there exists $\{q_{[\beta_n]-1}\}_{n=1}^{\infty}$ ($[x]$ denotes the largest integer less than or equal to x) with $q_{[\beta_n]-1} \in \mathcal{P}_{[\beta_n]-1}$ such that

$$\lim_{n\to\infty} (1+x)^{\beta_n - [\beta_n]+1} q_{[\beta_n]-1}(x) = r(x)$$

uniformly on $[-1, 1]$.

For $s(x)$, by Theorem 2.1 in [**11**] (or Theorem 1 in [**2**]), there exists

$$s_n(x) = \sum_{k=0}^{n-[\beta_n]+1} a_k^{(n)} x^k, \quad n = 1, 2, \ldots,$$

such that $\lim_{n\to\infty} (1+x)^{[\beta_n]-1} s_n(x) = s(x)$, uniformly on $[-1, 1]$. Hence

$$h(x) = r(x)s(x) = \lim_{n\to\infty} (1+x)^{\beta_n} q_{[\beta_n]-1}(x) s_n(x),$$

uniformly on $[-1, 1]$.

Note that, since $P_n(x) := q_{[\beta_n]-1}(x) \cdot s_n(x) \in \mathcal{P}_n$, $n = 1, 2, \dots$, we have proved that $h(x)$ and, hence, by Lemma 3.1, $f(x)$ is a uniform limit function of a sequence of weighted polynomials. □

Remark . The proof of (i) when $\theta_2 > 0$ can be obtained by several other methods. One of them is by slight modification of the proof of part (ii) given below. The other possibilities are suitable modifications of methods in [**2**] and [**11**].

Proof of (ii) *when* $\theta_1 > 0$ *and* $\theta_2 > 0$. The necessity follows from Corollary 2.6 in [**3**]. To prove the sufficiency, we define

(3.3)
$$E_{\alpha_n,\beta_n,n} := \inf\{\|(1-x)^{\alpha_n}(1+x)^{\beta_n}x^n - g(x)\|_I ; g \in (1-x)^{\alpha_n}(1+x)^{\beta_n}\mathcal{P}_{n-1}\},$$

where $\mathcal{P}_{n-1} := \{0\}$ if $n = 0$.

It is a consequence of Proposition 3.1 in [**3**] that there exists a unique $Q_n(x) := Q_{\alpha_n,\beta_n,n}(x) \in (1-x)^{\alpha_n}(1+x)^{\beta_n}\mathcal{P}_n$ satisfying

$$E_{\alpha_n,\beta_n,n} = \|Q_n\|_I = \|Q_{\alpha_n,\beta_n,n}\|_I. \tag{3.4}$$

Set

$$T_{\alpha_n,\beta_n,n}(z) := \frac{Q_n(z)}{E_{\alpha_n,\beta_n,n}}, \quad z \in \mathbf{C}. \tag{3.5}$$

It follows from Theorems 2.5 and 3.5 in [**3**] that

$$\lim_{n\to\infty} |T_{\alpha_n,\beta_n,n}(z)|^{\frac{1}{\alpha_n+\beta_n+n}} = G(z;\theta_1,\theta_2), \tag{3.6}$$

for $z \notin [a,b]$, where a, b are defined as in (2.2) and

(3.7)
$$G(z;\theta_1,\theta_2) := \begin{cases} |\varphi(z)| \left|\frac{\varphi(z)-\varphi(1)}{\varphi(1)\varphi(z)-1}\right|^{\tau_1} \left|\frac{\varphi(z)-\varphi(-1)}{\varphi(-1)\varphi(z)-1}\right|^{\tau_2}, & z \in \overline{\mathbf{C}}\backslash[a,b], \\ 1, & z \in [a,b], \end{cases}$$

with $\tau_1 := \theta_1/(1+\theta_1+\theta_2)$, $\tau_2 := \theta_2/(1+\theta_1+\theta_2)$ and

$$\varphi(z) = \varphi(z;\theta_1,\theta_2) := \frac{\sqrt{z-a}+\sqrt{z-b}}{\sqrt{z-a}-\sqrt{z-b}} \tag{3.8}$$

mapping $\overline{\mathbf{C}}\backslash[a,b]$ onto the exterior of the unit circle. Moreover, the extremal points (alternation points) of $T_{\alpha_n,\beta_n,n}(x)$ in $(-1,1)$ are dense in $[a,b]$ and

$$\lim_{n\to\infty}\xi_{1,n}=a, \quad \lim_{n\to\infty}\xi_{2,n}=b, \tag{3.9}$$

where $\xi_{1,n},\xi_{2,n}$ are the smallest and largest extremal points of $T_{\alpha_n,\beta_n,n}(x)$ in $(-1,1)$, respectively (cf. [**3**]).

Consider

$$g(z):=\varphi(z)\left(\frac{\varphi(z)-\varphi(1)}{1-\varphi(1)\varphi(z)}\right)^{\tau_1}\left(\frac{\varphi(z)-\varphi(-1)}{1-\varphi(-1)\varphi(z)}\right)^{\tau_2}.$$

By tedious but straightforward computation, we can get asymptotic expansions of $g(z)$ near $z=a$ and $z=b$:

$$g(z)=1+A_1(z-a)^{\frac{3}{2}}+A_2(z-a)^{\frac{4}{2}}+\cdots \tag{3.10}$$

and

$$g(z)=1+B_1(z-b)^{\frac{3}{2}}+B_2(z-b)^{\frac{4}{2}}+\cdots \tag{3.11}$$

with $A_1\neq 0$, $B_1\neq 0$.

By conformal mapping argument and the discussion about function $G(z;\theta_1,\theta_2)$ in [**3**] and (3.10), (3.11), one can show that the level curve $\{z:G(z;\theta_1,\theta_2)=1\}$ is of barbell shape and the ends are attached to the bar with angle (see Figure 1)

$$\Psi_a=\Psi_b=\frac{2\pi}{3}. \tag{3.12}$$

Our goal is to construct weighted polynomials such that

$$\lim_{n\to\infty}(1-x)^{\alpha_n}(1+x)^{\beta_n}P_n(x)=f(x), \tag{3.13}$$

where $f\in C(I)$ with $f(x)=0$ for $x\in[-1,a]\cup[b,1]$.

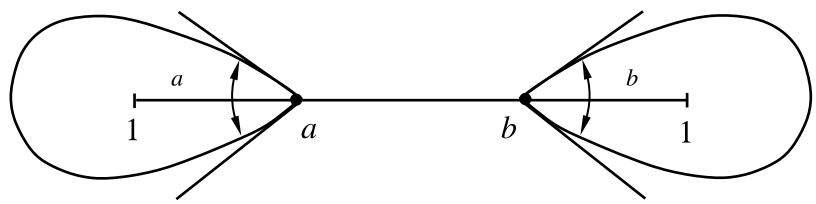

FIGURE 1.

To do this, we follow the method of Lubinsky and Saff [**6**] and first consider

$$h_n(z) = \begin{cases} p(z)(z-\xi_{1,n})(z-\xi_{2,n}), & \xi_{1,n} \le z \le \xi_{2,n}, \\ 0, & z \in [-1, \xi_{1,n}] \cup [\xi_{2,n}, 1], \end{cases}$$

where $\xi_{1,n}, \xi_{2,n}$ are as in (3.9), and $p(z)$ is an entire function.

We construct the polynomial $L_n(z) \in \mathcal{P}_{n-1}$ which interpolates $p(z)(z-\xi_{1,n})(z-\xi_{2,n})/(1-z)^{\alpha_n}(1+z)^{\beta_n}$ at the zeros of $\hat{Q}_n(z) := Q_n(z)/((1-z)^{\alpha_n}(1+z)^{\beta_n})$ in $[-1,1]$, where $Q_n(z)$ is defined as in (3.4). By the Hermite formula (cf. [**12**, p. 50]), we have

$$\text{(3.14)} \qquad \begin{aligned} &\frac{p(z)(z-\xi_{1,n})(z-\xi_{2,n})}{(1-z)^{\alpha_n}(1+z)^{\beta_n}} - L_n(z) \\ &= \frac{1}{2\pi i}\int_{\Gamma_n} \frac{p(\zeta)(\zeta-\xi_{1,n})(\zeta-\xi_{2,n})\hat{Q}_n(z)}{(\zeta-z)(1-\zeta)^{\alpha_n}(1+\zeta)^{\beta_n}\hat{Q}_n(\zeta)}\,d\zeta. \end{aligned}$$

The integral contour is chosen as follows: $\Gamma_n := \Gamma_{1,n} \cup \bar{\Gamma}_{1,n} \cup \Gamma_{2,n} \cup \Gamma_{3,n}$ oriented in a positive direction (see Figure 2).

We can write (3.14) as

$$\text{(3.15)} \qquad \begin{aligned} &h_n(z) - (1-z)^{\alpha_n}(1+z)^{\beta_n} L_n(z) \\ &= \frac{1}{2\pi i}\int_{\Gamma_n} \frac{p(\zeta)(\zeta-\xi_{1,n})(\zeta-\xi_{2,n})(1-z)^{\alpha_n}(1+z)^{\beta_n}\hat{Q}_n(z)}{(\zeta-z)(1-\zeta)^{\alpha_n}(1+\zeta)^{\beta_n}\hat{Q}_n(\zeta)}\,d\zeta \end{aligned}$$

for $z \in [\xi_{1,n}, \xi_{2,n}]$. In fact, it is easy to see, by explicitly writing

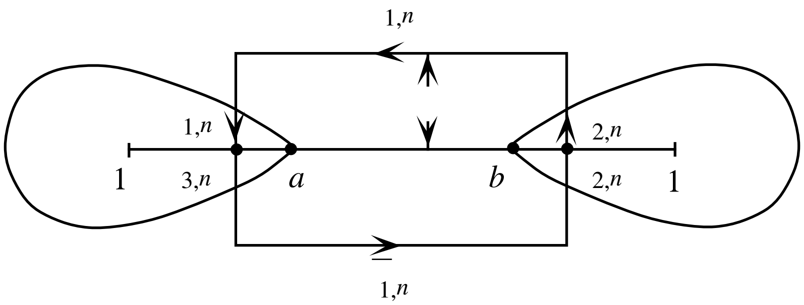

FIGURE 2.

$L_n(z)$ and using the residue theorem, that (3.15) remains true for all $z \in [-1, 1]$. We now estimate the error on I:

$$\begin{aligned}&\|h_n(z) - (1-z)^{\alpha_n}(1+z)^{\beta_n}L_n(z)\|_I \\ &\leq \frac{1}{2\pi}\|p\|_{\Gamma_n}\left\|\frac{(\zeta-\xi_{1,n})(\zeta-\xi_{2,n})}{\zeta - z}\right\|_{\substack{z\in I\\ \zeta\in\Gamma_n}}\int_{\Gamma_n}\frac{\|Q_n\|_I}{|Q_n(\zeta)|}\,|d\zeta|.\end{aligned} \tag{3.16}$$

It is easy to see from (3.16) that there exists $M = M(\delta)$, where δ denotes the height of $\Gamma_{1,n}$ above the real axis, such that

$$\|h_n(z) - (1-z)^{\alpha_n}(1+z)^{\beta_n}L_n(z)\|_I \leq M\int_{\Gamma_n}\frac{\|Q_n\|_I}{|Q_n(\zeta)|}\,|d\zeta|. \tag{3.17}$$

We claim that, for a suitable choice of $\delta > 0$,

$$\lim_{n\to\infty}\int_{\Gamma_n}\frac{\|Q_n\|_I}{|Q_n(\zeta)|}\,|d\zeta| = 0. \tag{3.18}$$

By (3.6), we have, for $z \notin [a, b]$,

$$\lim_{n\to\infty}\left\{\frac{\|Q_n\|_I}{|Q_n(z)|}\right\}^{\frac{1}{\alpha_n+\beta_n+n}} = \frac{1}{G(z;\theta_1,\theta_2)}.$$

By (3.12), for every $\delta > 0$, there exists $\rho := \rho(\delta) \in (0, 1)$, such that, for n large enough and all $z \in \Gamma_{1,n} \cup \bar{\Gamma}_{1,n}$, $G(z;\theta_1,\theta_2) > 1/\rho > 1$. Hence, for n large,

$$\int_{\Gamma_{1,n}\cup\bar{\Gamma}_{1,n}}\frac{\|Q_n\|_I}{|Q_n(\zeta)|}\,|d\zeta| \leq 4\rho^{n+\alpha_n+\beta_n} \leq \rho^n. \tag{3.19}$$

For the integral over $\Gamma_{2,n}$, we have

$$\begin{aligned}(3.20)\qquad &\int_{\Gamma_{2,n}} \frac{\|Q_n\|_I}{|Q_n(\zeta)|}\,|d\zeta| \\ &= \int_{\Gamma_{2,n}} \left|\frac{(1-\xi_{2,n})^{\alpha_n}(1+\xi_{2,n})^{\beta_n}}{(1-\zeta)^{\alpha_n}(1+\zeta)^{\beta_n}}\right| \left|\frac{\hat{Q}_n(\xi_{2,n})}{\hat{Q}_n(\zeta)}\right| |d\zeta|.\end{aligned}$$

First, it is easy to see that, for $\zeta \in \Gamma_{2,n}$,

$$(3.21)\qquad \left|\frac{(1-\xi_{2,n})^{\alpha_n}(1+\xi_{2,n})^{\beta_n}}{(1-\zeta)^{\alpha_n}(1+\zeta)^{\beta_n}}\right| \le 1.$$

Next, let $\zeta_{1,n}, \zeta_{2,n}, \dots, \zeta_{n,n}$ be the zeros of $\hat{Q}_n(\xi)$. Then, for $\zeta \in \Gamma_{2,n}$, $\zeta = \xi_{2,n} + iy$, $-\delta \le y \le \delta$, we have

$$\begin{aligned}\left|\frac{\hat{Q}_n(\xi_{2,n})}{\hat{Q}_n(\zeta)}\right| &= \prod_{j=1}^{n} |\xi_{2,n} - \zeta_{j,n}| \Big/ \prod_{j=1}^{n} |\xi_{2,n} + iy - \zeta_{j,n}| \\ &= \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \log\left[1 + \frac{y^2}{(\xi_{2,n} - \zeta_{j,n})^2}\right]\right\} \\ &\le \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \log\left[1 + \frac{y^2}{(b+\varepsilon - \zeta_{j,n})^2}\right]\right\}\end{aligned}$$

for every $\varepsilon > 0$ and n sufficiently large. Choosing $\delta = \varepsilon/2$, and noting that $\log(1+x) \ge x/2$ for $0 \le x \le 1/2$, we have, for every $\varepsilon > 0$,

$$(3.22)\qquad \left|\frac{\hat{Q}_n(\xi_{2,n})}{\hat{Q}_n(\zeta)}\right| \le \exp\left\{-\frac{1}{4}y^2 \sum_{j=1}^{n} \frac{1}{(b+\varepsilon-\zeta_{j,n})^2}\right\},$$

for $\zeta \in \Gamma_{2,n}$ and n large enough.

It is a consequence of Theorem 3.4 in [**10**] that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} &\frac{1}{(b+\varepsilon-\zeta_{j,n})^2} \\ &= \int_a^b \frac{1}{(b+\varepsilon-t)^2} \cdot \frac{\sqrt{(b-t)(t-a)}}{\pi(1-t^2)}\,dt.\end{aligned}$$

Notice that

$$\lim_{\varepsilon\to 0}\int_a^b \frac{1}{(b+\varepsilon-t)^2}\cdot\frac{\sqrt{(b-t)(t-a)}}{\pi(1-t^2)}\,dt=\infty,$$

and so, given $\sigma>0$, we can find $\varepsilon=\varepsilon(\sigma)>0$ and $n(\sigma)$ such that

$$\exp\left\{-\frac{1}{4}y^2\sum_{j=1}^{n}\frac{1}{(b+\varepsilon-\zeta_{j,n})^2}\right\}\le \exp\left\{-y^2\frac{n}{\sigma^2}\right\}, \tag{3.23}$$

for $n\ge n(\sigma)$.

Combining (3.20)–(3.23), we obtain

$$\int_{\Gamma_{2,n}}\frac{\|Q_n\|_I}{|Q_n(\zeta)|}|d\zeta|\le\int_{-\delta}^{\delta}\exp\left\{-y^2\frac{n}{\sigma^2}\right\}\,dy\le\pi\sigma n^{-\frac{1}{2}}, \tag{3.24}$$

for $n\ge n(\sigma)$. Similarly, we can get

$$\int_{\Gamma_{3,n}}\frac{\|Q_n\|_I}{|Q_n(\zeta)|}|d\zeta|\le\pi\sigma n^{-\frac{1}{2}}\quad\text{for } n\ge n(\sigma). \tag{3.25}$$

It follows from (3.19), (3.24) and (3.25) that our claim (3.18) is true. Now consider

$$h(z):=\begin{cases} p(z)(z-a)(z-b), & z\in[a,b],\\ 0, & z\notin[a,b],\end{cases} \tag{3.26}$$

where $p(z)$ is an entire function. Then it is easy to see from (3.9) that $\lim_{n\to\infty}h_n(z)=h(z)$ uniformly on I. Hence, $h(z)$ is the uniform limit on I of the sequence $(1-z)^{\alpha_n}(1+z)^{\beta_n}L_n(z)$. Since any function $f\in C(I)$ with $f(x)=0$ for $x\in[-1,a]\cup[b,1]$ is a limit function of functions having a form like $h(x)$ defined in (3.26) in the case $\theta_1,\theta_2\ne\infty$ (this implies $a<b$) by Lemma 3.1 we have proved (3.13).

In the case when $\theta_1=\infty$ and/or $\theta_2=\infty$, we get $a=b$ and $\{f\in C(I): f(x)=0 \text{ for } x\in[-1,a]\cup[b,1]\}$ consists only of the identically zero constant. Obviously, part (ii) of Theorem 1 is true in this case.

The proof for the case when only one of θ_1 and θ_2 is zero is contained in the proof of (iii).

Proof of (ii) *when* $\theta_1 = \theta_2 = 0$. We only need to prove the sufficiency. Assume $f \in C(I)$ and $f(1) = f(-1) = 0$; then, as usual, we can uniformly approximate f by polynomials of the form $(1-x)(1+x)p_m(x)$, where $p_m \in \mathcal{P}_m$, $m = 1, 2, \dots$. Hence, it is enough to show that we can uniformly approximate $(1-x)(1+x)p_m(x)$ for every $p_m \in \mathcal{P}_m$, $m = 1, 2, \dots$.

Let

$$f(x) = (1-x)(1+x)p_m(x);$$

then, by part (i) of Theorem 1, $(1+x)p_m(x)$ can be uniformly approximated by a sequence of polynomials $\{(1+x)^{\beta_n} q_n(x)\}_{n=1}^{\infty}$, where $q_n \in p_{[\frac{n}{2}]}$. On the other hand, using part (i) again, $(1-x)$ can be uniformly approximated by another sequence of polynomials $\{(1-x)^{\alpha_n}\hat{q}_n(x)\}_{n=1}^{\infty}$, where $\hat{q}_n(x) \in p_{[\frac{n}{2}]}$. Therefore, $(1-x)(1+x)p_m(x)$ is the uniform limit of the sequence of the form

$$\{(1-x)^{\alpha_n}(1+x)^{\beta_n} q_n(x)\hat{q}_n(x)\}_{n=1}^{\infty}$$

and $q_n\hat{q}_n \in \mathcal{P}_n$. This completes the proof of part (ii).

Proof of (iii). The necessity of (iii) is an easy consequence of Theorem 2.2 in [**11**] or Corollary 2.6 in [**3**]. We now prove the sufficiency. Let us first assume $\alpha_n \ge 1$ for n large enough. It suffices to prove that one can uniformly approximate a function

$$h(x) = (1-x)r(x),$$

where

$$r(x) = \begin{cases} (x-a)q(x), & x \in [a,1], \\ 0, & x \in [-1,a], \end{cases}$$

and $q \in \mathcal{P}_n$, $n = 1, 2, \dots$.

We claim that we can find two sequences of polynomials $\{p_n(x)\}_{n=1}^{\infty}$ and $\{q_n(x)\}_{n=1}^{\infty}$ with $p_n, q_n \in \mathcal{P}_n$ such that

$$\lim_{n\to\infty} (1+x)^{\beta_n + [\sqrt{n\alpha_n}]+1} p_n(x) = r(x) \tag{3.27}$$

uniformly on I and

$$\lim_{n\to\infty}(1-x)^{\alpha_n}q_{[n\alpha_n]+1}(x)=(1-x) \tag{3.28}$$

uniformly on I. In fact, since

$$\lim_{n\to\infty}\frac{\beta_{n+[\sqrt{n\alpha_n}]+1}}{n}=\lim_{n\to\infty}\frac{\beta_n}{n}\lim_{n\to\infty}\frac{n+[\sqrt{n\alpha_n}]+1}{n}=\theta_2,$$

we know that (3.27) is possible by part (i) of Theorem 1.

For (3.28) we need a little more effort. First note that

$$0\le\varliminf_{n\to\infty}\frac{\alpha_n}{[\sqrt{n\alpha_n}]+1}\le\varlimsup_{n\to\infty}\frac{\alpha_n}{[\sqrt{n\alpha_n}]+1}\le\lim_{n\to\infty}\frac{\alpha_n}{\sqrt{n\alpha_n}}=0,$$

so

$$\lim_{n\to\infty}\frac{\alpha_n}{[\sqrt{n\alpha_n}]+1}=0.$$

Second, since we assume $\alpha_n\ge1$ for n large enough, we have

$$\lim_{n\to\infty}([\sqrt{n\alpha_n}]+1)=\infty.$$

Similar to the proof of (ii) when $\theta_2=0$, we can show that there exists $q_{[\sqrt{n\alpha_n}]+1}$ such that (3.28) holds.

It follows from (3.27) and (3.28) that

$$\lim_{n\to\infty}(1-x)^{\alpha_n}(1+x)^{\beta_n}p_{n-[\sqrt{n\alpha_n}]-1}(x)q_{[\sqrt{n\alpha_n}]+1}(x)=h(x)$$

uniformly on I.

Now for the remaining case when $0\le\alpha_n<1$ for infinitely many n, by the above discussion, there exists $q_n\in\mathcal{P}_{n-1}$, $n=1,2,3,\ldots$, such that

$$\lim_{n\to\infty}(1-x)^{\alpha_n+1}(1+x)^{\beta_n}q_n(x)=\lim_{n\to\infty}(1-x)^{\alpha_n}(1+x)^{\beta_n}p_n(x)=f(x),$$

uniformly on I, where $p_n(x):=(1-x)q_n(x)\in\mathcal{P}_n$.

Proof of (iv). This is an easy consequence of (ii). We omit the details. □

Proof of Theorem 2. First we consider the case when f is analytic on $[c,d]$.

For c,d given, we consider the extremal

$$\inf_{\substack{p_n \in \mathcal{P}_n \\ p_n(x)=x^n+\cdots}} \|(1-x)^{n\theta_1}(1+x)^{n\theta_2}p_n\|_{[c,d]} =: G_n.$$

We know that there are $T_n \in \mathcal{P}_n$, $n=1,2,\dots$, $T_n(x)=x^n+\cdots$ such that

$$\|(1-x)^{n\theta_1}(1+x)^{n\theta_2}T_n\|_{[c,d]} = G_n.$$

Now we can employ the notation and results in [**8**] (particularly Theorem 2.3). To do so, we take $[c,d]$ as Σ and $(1-x)^{\theta_1}(1+x)^{\theta_2}$ as $w(x)$. Then it is easy to see that $\mathcal{S}=[c,d]$ and there exists a unique $\mu^* \in \mathcal{M}([c,d])$. ($\mathcal{M}([c,d])$ denotes the collection of all positive unit Borel measures μ with $\operatorname{supp}(\mu) \subset [c,d]$) such that

$$\int_c^d \log|x-t|\,d\mu^*(t) = -\log|(1-x)^{\theta_1}(1+x)^{\theta_2}| + F \tag{3.29}$$

for $x \in [c,d]$, where

$$F := \log\frac{d-c}{4} + \int_c^d \frac{\log(1-x)^{\theta_1}(1+x)^{\theta_2}}{\pi\sqrt{(x-c)(d-x)}}\,dx.$$

Moreover, we have

$$\lim_{n\to\infty} G_n^{1/n} = \exp(F), \tag{3.30}$$

and

$$\lim_{n\to\infty} |(1-z)^{n\theta_1}(1+z)^{n\theta_2}T_n(z)|^{\frac{1}{n}} = \exp(F)\exp V(z) \tag{3.31}$$

locally uniformly for $z \in \mathbf{C}\backslash([c,d]\cup\{1,-1\})$, where

$$V(z) := \int_c^d \log|z-t|\,d\mu^*(t) + \log|(1-z)^{\theta_1}(1+z)^{\theta_2}| - F \tag{3.32}$$

for $z \in \mathbf{C}$.

In order to prove our theorem, we need the following lemmas.

Lemma 3.2. *For $z := x + iy$, $x \in [c,d]$, $y > 0$, $V(x+iy)$ is an increasing function of y. Similarly, for $y < 0$, $V(x+iy)$ is a decreasing function of y.*

Proof. It is obvious that

$$\frac{d}{dy}V(x+iy) = \int_c^d \frac{y}{(x-t)^2+y^2}\,d\mu^*(t) + \frac{\theta_1 y}{(x-1)^2+y^2} + \frac{\theta_2 y}{(x+1)^2+y^2} > 0, \quad \text{for } y > 0. \quad \square$$

Lemma 3.3. *There exist r_1, r_2 such that $a < r_1 < c < d < r_2 < b$ and $V(r_1) > 0$, $V(r_2) > 0$. Furthermore, r_1, r_2 can be chosen such that $V(x)$ is decreasing in $[r_1, c]$ and increasing in $[d, r_2]$.*

Proof. If there is no r_1 in (a,c) such that $V(r_1) > 0$, then, since

$$|(1-z)^{n\theta_1}(1+z)^{n\theta_2}p_n(z)| \le \|(1-z)^{n\theta_1}(1+z)^{n\theta_2}p_n(z)\|_{[c,d]}e^{nV(z)}$$

for all $z \in \mathbf{C}$, we have, in particular,

$$|(1-x)^{n\theta_1}(1+x)^{n\theta_2}p_n(x)| < \|(1-x)^{n\theta_1}(1+x)^{n\theta_2}p_n(x)\|_{[c,d]}$$

for $x \in [a,c]$. This means that, for each $n = 0,1,2,\dots, p_n \in \mathcal{P}_n$,

$$\|(1-x)^{n\theta_1}(1+x)^{n\theta_2}p_n(x)\|_{[-1,1]} = \|(1-x)^{n\theta_1}(1+x)^{n\theta_2}p_n(x)\|_{[c,b]},$$

which contradicts the definition of a and b. Thus, we proved the existence of r_1. Similarly, we can establish the existence of $r_2 \in (d,b)$ such that $V(r_2) > 0$.

Now choose r_1 satisfying $V(r_1) > 0$ and

$$r_1 = \max\{r : V'(r) = 0,\ r \in [a,c]\}.$$

If $V(x)$ is not decreasing in $[r_1, c]$, there must exist $\hat{r}_1 \in (r_1, c)$ such that

$$V(x) < 0 \quad \text{for all } x \in (\hat{r}_1, c).$$

Thus, for $\Sigma_1 := [r_1, d]$ and $w(x) = (1-x)^{\theta_1}(1+x)^{\theta_2}$,

$$\|w^n p_n\|_{\Sigma_1} = \|w^n p_n\|_{[c,d]}$$

for all $p_n \in \mathcal{P}_n$. But this is impossible since we can easily prove that the subinterval where the sup norm lives should be the whole interval $[r_1, d]$ as we showed for $[c, d]$ at the beginning of our proof. So $V(x)$ is decreasing in $[r_1, c]$.

By the same reasoning we can show that r_2 can be chosen such that $V(x)$ is increasing in (d, r_2). □

Lemma 3.4. *For all $\delta > 0$ small enough, the level curve $\Gamma_\delta : V(z) = \delta$ has a component which is a loop surrounding $[c, d]$ but not -1 and 1; with $\delta \to 0^+$ this loop shrinks to $[c, d]$.*

Proof. This is a consequence of the implicit function theorem and above lemmas. □

We now return to the proof of Theorem 2. Since f is analytic on $[c, d]$, by Lemma 3.4, there is a $\delta_0 > 0$ such that f is analytic inside and on Γ_δ (we still use Γ_δ to denote the loop in Lemma 3.4) for $\delta < \delta_0$. Now let $z_1, z_2, \dots, z_n$ be the n zeros of T_n in $[c, d]$, and let $L_n \in \mathcal{P}_{n-1}$ satisfy

$$L_n(z_j) = f(z_j)/(1-z_j)^{n\theta_1}(1+z_j)^{n\theta_2}, \quad j = 1, 2, \dots, n.$$

Then, by Hermite's formula,

$$\frac{f(x)}{(1-x)^{n\theta_1}(1+x)^{n\theta_2}} - L_n(x) = \frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{T_n(x) f(t)\, dt}{(1-t)^{n\theta_1}(1+t)^{n\theta_2} T_n(t)(t-x)}$$

for $x \in [c,d]$, and so

$$\begin{aligned}
&|f(x) - (1-x)^{n\theta_1}(1+x)^{n\theta_2}L_n(x)| \\
&\quad \le \frac{1}{2\pi}\int_{\Gamma_\delta} \frac{|(1-x)^{n\theta_1}(1+x)^{n\theta_2}T_n(x)||f(t)|}{|(1-t)^{n\theta_1}(1+t)^{n\theta_2}T_n(t)||t-x|}\,|dt| \\
&\quad \le \frac{1}{2\pi}\frac{G_n \cdot \max_{t\in\Gamma_\delta}|f(t)|\cdot\gamma(\Gamma_\delta)}{(e^F\cdot e^\delta - \varepsilon)^n \operatorname{dist}(\Gamma_\delta,[c,d])} \\
&\quad \le K_1 e^{-\delta_n}\max_{t\in\Gamma_\delta}|f(t)| = O(e^{-\delta n}) \quad \text{as } n\to\infty,
\end{aligned} \tag{3.33}$$

where $\gamma(\Gamma_\delta)$ denotes the length of Γ_δ. So (2.5) holds.

On the other hand, if (2.5) holds, it follows from Lemma 3.4 and the inequality $\|(1-z)^{n\theta_1}(1+z)^{n\theta_2}p_n(z)\|_{\Gamma_\delta} \le \|\cdot\|_{[c,d]}\cdot e^{\delta n}$ that, when δ is small enough,

$$\limsup_{n\to\infty}\|(1-z)^{n\theta_1}(1+z)^{n\theta_2}p_n(z)\|_{\Gamma_\delta}^{\frac{1}{n}} < 1.$$

So, f can have an analytic continuation in $\operatorname{Int}(\Gamma_\delta)$.

Next, if f is a continuous function, then there exist polynomials $\{p_n^*\}$ such that (by Jackson's Theorem, [**1**, p. 144])

$$\|f - p_n^*\|_{[c,d]} \le K\omega\left(f, \frac{1}{n}\right).$$

Take $\delta > 0$ such that Γ_δ is a loop surrounding $[c,d]$ but not -1 and 1, and take $\sigma > 1$ such that Γ_δ is contained in the region bounded by the ellipse E_σ: $|z-c|+|z-d| = (d-c)\sigma$. Let $\lambda = \delta/(2\ln\sigma)$, for $n > 2/\lambda$, and let $L_n^* \in \mathcal{P}_{n-1}$ such that

$$L_n^*(z_j) = \frac{p_{[\lambda n]}^*(z_j)}{(1-z_j)^{n\theta_1}(1+z_j)^{n\theta_2}}, \quad j = 1,2,\ldots,n,$$

where z_j, $j = 1,2,\ldots,n$, are the zeros of T_n. Then we have

$$\begin{aligned}
&|f(x) - (1-x)^{n\theta_1}(1+x)^{n\theta_2}L_n^*(x)| \\
&\quad \le |f(x) - p_{[\lambda n]}^*(x)| + |p_{[\lambda n]}^*(x) - (1-x)^{n\theta_1}(1+x)^{n\theta_2}L_n^*(x)| \\
&\quad \le K\omega\left(f, \frac{1}{\lambda n - 1}\right) + K_1 e^{-\delta n}\max_{x\in\Gamma_\delta} p_{[\lambda n]}^*(x) \\
&\quad \le K_2\omega\left(f, \frac{1}{n}\right) + K_3 e^{-\delta n}\sigma^{\lambda n}\cdot 2\|f\|_{[c,d]},
\end{aligned}$$

where we have used (3.33) and the Bernstein Lemma (cf. [**12**, p. 77]). So, by the choice of λ, we have

$$(3.34)\quad |f(x) - (1-x)^{n\theta_1}(1+x)^{n\theta_2}L_n^*(x)| \le K_2\omega\left(f, \frac{1}{n}\right) + K_4 e^{-\tau n},$$

for some $\tau > 0$, where K_2, τ can be chosen dependent only on c, d, θ_1 and θ_2. Taking K_4 large enough, we see that (2.4) holds for all n. □

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