9-1-1997

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SIAM J. MATH. ANAL.
Vol. 28, No. 5, pp. 1248–1257, September 1997

ASYMPTOTICS OF THE ZEROS OF RELATIVISTIC HERMITE POLYNOMIALS*

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Abstract. The relativistic Hermite polynomial (RHP) is a class of orthogonal polynomials associated with varying weights. We study the asymptotics of the zeros of the RHP when both degree n of polynomials and relativistic parameter N approach infinity.

Key words. Hermite polynomials, orthogonal polynomials

AMS subject classifications. 41A, 41C

PII. S003614109529642X

1. Introduction.

Relativistic Hermite polynomials (RHPs) $\{H_n^{(N)}(x)\}_{n=0}^{\infty}$ were introduced in [1] in connection with the wave functions of the quantum relativistic harmonic oscillator. It was shown in [1] that the RHP satisfies the second-order differential equation

$$\left(1+\frac{x^2}{N}\right)y_n'' - \frac{2}{N}(N+n-1)xy_n' + \frac{n}{N}(2N+n-1)y_n = 0. \tag{1.1}$$

Equation (1.1) is a particular case of a second-order hypergeometric-type equation [9]

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0, \tag{1.2}$$

where

$$\sigma(x) = \left(1+\frac{x^2}{N}\right),$$

$$\tau = -\frac{2}{N}(N+n-1)x,$$

$$\lambda = \frac{n}{N}(2N+n-1).$$

It is easy to verify that the following relation holds:

$$\lambda = -n\tau' - \frac{1}{2}n(n-1)\sigma''.$$

By solving the equation

$$[\sigma(x)\rho_n(x;N)]' = \tau(x)\rho_n(x;N),$$

*Received by the editors December 18, 1995; accepted for publication April 17, 1996.
http://www.siam.org/journals/sima/28-5/29642.hmtl

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one can find the symmetric factor or weight function

$$\rho_n(x;N) = \left(1+\frac{x^2}{N}\right)^{-(N+n)}, \quad N > \frac{1}{2}, \quad n = 0,1,2,\ldots,\ x \in (-\infty,\infty). \tag{1.3}$$

Using this weight function, the following orthogonality of the RHP was established in [1]:

$$\int_{-\infty}^{\infty} x^k H_n^{(N)}(x)\rho_n(x;N)dx = 0, \qquad k = 0,1,\ldots,n-1. \tag{1.4}$$

That is, $\{H_n^{(N)}(x)\}_{n=0}^{\infty}$ is a class of orthogonal polynomials with respect to a sequence of varying weight functions $\rho_n(x)$. Clearly,

$$\lim_{N\to\infty} \rho_n(x;N) = e^{-x^2}$$

and

$$\lim_{N\to\infty} H_n^{(N)}(x) = H_n(x).$$

So the relativistic Hermite polynomials become classical Hermite polynomials when the relativistic parameter $N \to \infty$.

The distributions of zeros of RHP were studied in [2]. An analytic approximation for the distribution was derived within the framework of the WKB approximation.

The asymptotics of orthogonal polynomials with respect to varying weights are closely related to constrained or weighted polynomial approximation. Logarithmic potential has been extensively used in investigating such asymptotics. We study the asymptotics of the zeros of RHP when both n and N approach ∞ by using the potential-theoretic method.

The paper is organized as follows: in order to state our main results, we shall introduce some basics from potential theory in section 2. Applying a general result from potential theory developed in [11] to our relativistic weight function, we determine the support of the equilibrium measure explicitly in section 3. In section 4, we give an explicit formula for the density function of the equilibrium measure. The asymptotics of the zeros of the RHP when both n and N approach ∞ are determined in section 5.

2. Basics of potential theory. We shall use logarithmic potentials of Borel measures. If μ is a finite Borel measure with compact support, then its logarithmic potential is defined as its convolution with the logarithmic kernel:

$$U^{\mu}(z) = \int \log\frac{1}{|z-t|}d\mu(t).$$

Let E be a closed subset of the real number line. A weight function w on E is said to be admissible if it satisfies the following three conditions:

(i) w is continuous;

(ii) $\mathrm{Cap}\{x \in E \mid w(x) > 0\} > 0$;

(iii) $Z := \{x \in E : w(x) = 0\}$ has capacity zero; and

(iv) if E is unbounded, then $|x|w(x) \to 0$ as $|x| \to \infty$, $x \in E$.

We say that w is strongly admissible if

(i) w^q is admissible for every q, $0 < q \le 1$;

(ii) E is regular, i.e., for all k large, $E \cap [-k,k]$ is regular with respect to the Dirichlet problem for its complement on the Riemann sphere, and

(iii) $E \backslash Z$ is interval-like.

We define $Q = Q_w$ by

$$w(x) = \exp(-Q(x)). \tag{2.1}$$

Then $Q : E \to (-\infty, \infty]$ is continuous everywhere where w is positive, i.e., Q is finite.

Let $\mathcal{M}(E)$ be the set of all positive unit Borel measures μ with support $S(\mu) := \mathrm{supp}(\mu) \subset E$ and define the weighted energy integral

$$I_w[\mu] = \iint \log \frac{1}{|z-t|w(z)w(t)} d\mu(z)d\mu(t).$$

Let

$$V_w(E) = \inf\{I_w[\mu] \mid \mu \in \mathcal{M}(E)\}.$$

Then the following properties are true (cf. [12], [10]).

(i) $V_w(E)$ is finite.

(ii) There exists a unique $\mu_E \in \mathcal{M}(E)$ such that

$$I_w(\mu_w) = V_w(E).$$

Moreover, μ_w has finite logarithmic energy.

(iii) $S(\mu_w)$ is a compact subset of E.

(iv) The inequality

$$U^{\mu_w}(z) + Q(z) \geq F_w, \quad z \in E. \tag{2.2}$$

(v) The equality

$$U^{\mu_w}(z) + Q(z) = F_w, \quad z \in S(\mu_w). \tag{2.3}$$

The measure μ_w is called the equilibrium or extremal measure in the presence of an external field, and

$$F_w = V_w(E) - \int Q d\mu_w. \tag{2.4}$$

In order to state our applications to polynomial extremal problems, we define

$$E_{n,p}(w) := \inf\{\|[w(x)]^n[x^n - P(x)]\|_{E,p} : P \in \mathcal{P}_{n-1}\},$$

where $\mathcal{P}_n$ is the set of all polynomials with degrees $\leq n$ and

$$\|f\|_{E,p} := \left(\int_E |f|^p dx\right)^{1/p},$$

$n = 1, 2, \dots, 0 < p \leq \infty$. The extremal polynomials $T_n(x; w, p) = x^n + \cdots \in \mathcal{P}_n$ are defined by the property

$$E_{n,p}(x) = \|[w(x)]^n T_n(x; w, p)\|_{E,p}.$$

Finally, in this section, we state the following two lemmas.

LEMMA 2.1 (see [6]). *Let w be strongly admissible and $0 < p \leq \infty$. Let $\{t_{n,k}\}_{k=1}^{n}$ be the zeros of $T_n(x; w, p)$. Then there exists a closed bounded interval I containing $S(\mu_w)$ and all the zeros of $T_n(x; w, p)$. Moreover,*

$$\lim_{n\to\infty} |T_n(x; w, p)|^{1/n} = \exp\left[\int \log|z-t| d\mu_w(t)\right]$$

uniformly on every compact set of the complex plane disjoint from I,

$$\lim_{n\to\infty} [E_{n,p}]^{1/n} = \exp(F_w),$$

and

$$\lim_{n\to\infty} \mu_n = \mu_w$$

in the weak-star topology, where

$$\mu_n(B) := \frac{1}{n}\#\{k : t_{n,k} \in B\}, \qquad n = 1, 2, \dots,$$

for any Borel set B.

LEMMA 2.2 (see [6]). *Let w be strongly admissible and $0 < p \leq \infty$. Suppose that $I \subset \mathbf{R}$ is a closed bounded interval containing $S(\mu_w)$. Let $\{v_{n,k}\}_{k=1}^{n}$ be a triangular scheme of points lying in I. With this scheme, let $q_n(x) = \prod_{k=1}^{n}(x - v_{n,k})$. Assume that for some p $(0 < p \leq \infty)$,*

$$\lim_{n\to\infty} \|w^n q_n\|_{E,p}^{1/n} \leq \exp(F_w).$$

Then

$$\lim_{n\to\infty} |q_n(x)|^{1/n} = \exp\left[\int \log|z-t| d\mu_w(t)\right]$$

uniformly on every compact set of the complex plane disjoint from I, and

$$\lim_{n\to\infty} \mu_n = \mu_w$$

in the weak-star topology, where

$$\mu_n(B) := \frac{1}{n}|\{k : v_{n,k} \in B\}|, \qquad n = 1, 2, \dots,$$

for any Borel set B.

3. Support of equilibrium measure. A fundamental theorem [5] in weighted polynomial approximation asserts that every weighted polynomial $\{w^n(x)p_n(x)\}$ must assume its maximum modulus on $S(\mu)$, i.e.,

$$\|w^n(x)p_n(x)\|_E = \|w^n(x)p_n(x)\|_{S(\mu)}, \tag{3.1}$$

where $S(\mu)$ is the support of the equilibrium measure of the set E, and $\|\cdot\|_E$ is the sup norm.

In this section we determine explicitly the support of the equilibrium measure $S(\mu)$ for the weight function $\rho_n(x;N)$. To find $S(\mu)$, we shall need to directly maximize the following F-functional [5]:

$$F(a,b) = \log\left(\frac{(b-a)}{4}\right) - \frac{1}{\pi}\int_a^b \frac{Q(x)}{\sqrt{(x-a)(b-x)}}\,dx.$$

We define $w_n(x) = \rho_n^{\frac{1}{2n}}(x;N)$. Then we have

$$Q_n(x) = \log\frac{1}{w_n(x)} = \left(\frac{1}{2}+\frac{N}{2n}\right)\log\left(1+\frac{x^2}{N}\right).$$

Since $Q_n(x)$ is an even function, the F-functional can be written as follows:

$$F(a) := f(-a,a) = \log a - \frac{1}{\pi}\left(1+\frac{N}{n}\right)\int_0^a \frac{\log\left(1+\frac{t^2}{N}\right)}{\sqrt{a^2-t^2}}\,dt - \log 2.$$

By an elementary integral formula [3],

$$\int_0^1 \frac{\log(1+bx^2)}{\sqrt{1-x^2}}\,ds = \pi\log\frac{1+\sqrt{1+b}}{2}.$$

We have

$$F(a) = \log\frac{a}{2} - \left(1+\frac{N}{n}\right)\log\frac{1+\sqrt{1+\frac{a^2}{N}}}{2}.$$

It is now elementary to check that the choice of $a = a_n$, which maximizes $F(-a,a)$, is given by

$$a_n = \sqrt{\frac{n(n+2N)}{N}}. \tag{3.2}$$

Therefore, we have determined the support $[-a_n, a_n]$ of equilibrium measure corresponding to varying weight $\rho_n(x;N)$.

Furthermore, we can determine the constant F_{w_n},

$$F_{w_n} = \log\frac{a_n}{2} - \left(1+\frac{N}{n}\right)\log\frac{1+\sqrt{1+\frac{a_n^2}{N}}}{2}.$$

We note that

$$\lim_{N\to\infty} a_n = \sqrt{2n},$$

$$\lim_{n\to\infty} F_{w_n} = -\frac{1}{2}\log N,$$

which concides with the results of [8]. We remark here that, although we use a potential-theoretic approach similar to the one used in [8], our approach is more direct. We shall continue our investigation along the same direction to determine the equilibrium measure.

4. Equilibrium measure. In section 3, we determined the support $S(\mu_n) = [-a_n, a_n]$ of equilibrium measure μ_n associated with varying weight $w_n(x)$. In this section, we apply a general formula [12, p. 53] for the density function of the equilibrium measure to our weight function $w_n(x)$ and find the following theorem.

THEOREM 4.1.

$$d\mu_n(t) = g_n(t)dt = \frac{N}{n\pi}\,\frac{\sqrt{a_n^2 - t^2}}{N + t^2}\,dt, \qquad t \in S(\mu_n). \tag{4.1}$$

Proof. It was shown in [12, Lem. 5.1] that the integral equation

$$\int_{-1}^{1} \log \frac{1}{|x-t|} g(t)dt = -Q(x) + C,$$

where C is some constant, has a solution $g(t)$ of the form

$$g(t) = \frac{2}{\pi^2}\sqrt{1-t^2}\int_0^1 \frac{sQ'(s) - tQ'(t)}{(1-s^2)^{1/2}(s^2-t^2)}\,ds + \frac{D_1}{\sqrt{1-t^2}}, \tag{4.2}$$

where

$$D_1 = \frac{1}{\pi} - \frac{1}{\pi^2}\int_{-1}^{1} \frac{sQ'(s)}{\sqrt{1-s^2}}\,ds. \tag{4.3}$$

$g(t)$ is even and has total integral 1 over $[-1, 1]$. Apply (4.2) and (4.3) to

$$Q_n(a_n x) = \left(\frac{1}{2} + \frac{N}{2n}\right)\log\left(1 + \frac{a_n^2 x^2}{N}\right),$$

and we get

$$g_n(t) = \frac{N}{n\pi}\,\frac{\sqrt{a_n^2 - t^2}}{N + t^2}.$$

We note that

$$\lim_{n\to\infty} g_n(t) = \frac{\sqrt{N}}{\pi(N + t^2)}, \qquad t \in (-\infty, \infty). \square$$

5. Asymptotics of zeros. In this section, we study the zeros distribution of $H_n^{(N)}(x)$ for n, $N \to \infty$. The following lemma tells us that the support of $H_n^{(N)}(x)$ "lives" also in some compact set in L_2.

LEMMA 5.3. *For $w(x) = (1 + x^2/N)^{(-N-n)/2}$, there is a positive constant A independent of n, N, such that, for $p \in \mathcal{P}_n$,*

$$\|w(x)p(x)\|_{(-\infty,\infty),2} \le 2\|w(x)p(x)\|_{[-Aa_n, Aa_n],2}.$$

Proof. The proof can be found in [4] for the fixed weight. Here we have a varying weight, so we may proceed exactly as in Theorem 5.2 in [4] to get the lemma. $\square$

The next theorem will discuss the location of the zeros of $H_n^{(N)}(x)$.

THEOREM 5.2. *For $w(x) = (1 + x^2/N)^{(-N-n)/2}$, there is a positive constant D independent of n, N, such that all the zeros of $H_n^{(N)}(x)$ lie in $[-Da_n, Da_n]$.*

Proof. Let $X_{n,N}$ denote the largest zero of $H_n^{(N)}(x)$. Suppose now that $\forall A > 0$ there exist n, N such that $X_{n,N} > Aa_n$. Let

$$t_{n,N}(x) := \frac{x - Aa_n}{x - X_{n,N}} H_n^{(N)}(x).$$

Then, for $x \in [-Aa_n, Aa_n]$,

$$|t_{n,N}(x)| \leq \frac{2Aa_n}{X_{n,N} - Aa_n} |H_n^{(N)}(x)|.$$

Hence, from the lemma above, we have

$$\begin{aligned} &\|w(x)t_{n,N}(x)\|_{(-\infty,\infty),2} \\ &\quad \leq 2\|w(x)t_{n,N}(x)\|_{[-Aa_n,Aa_n],2} \\ &\quad \leq 2\left(\frac{2Aa_n}{X_{n,N} - Aa_{n,N}}\right)\|w(x)H_n^{(N)}(x)\|_{[-Aa_n,Aa_n],2} \\ &\quad \leq 2\left(\frac{2Aa_n}{X_{n,N} - Aa_{n,N}}\right)\|w(x)H_n^{(N)}(x)\|_{(-\infty,\infty),2}. \end{aligned}$$

Thus, since $H_n^{(N)}(x)$ is extremal, the inequality implies that $1 \leq 4Aa_n/(X_{n,N} - Aa_n)$, that is, $X_{n,N} \leq 5Aa_n$. □

Now, we consider the case in which both n and N converge to ∞ with the same rate.

THEOREM 5.3. *Let $N = \lambda n$ and a_n be as in (3.2), where λ is a fixed number. Then*

$$\lim_{n\to\infty} |H_n^{(N)}(a_n x)|^{\frac{1}{n}} = \exp\left[\int_{-1}^{1} \log|z - t| d\mu(t)\right]$$

locally uniformly in $\mathbf{C}\backslash[-1, 1]$, where

$$d\mu(t) = \frac{\lambda(1 + 2\lambda)}{\pi} \quad \frac{\sqrt{1 - t^2}}{\lambda^2 + (1 + 2\lambda)t^2}\, dt.$$

Furthermore, let $\{t_{n,k}\}_{k=1}^{n}$ be the zeros of $H_n^{(N)}(a_n x)$ and B be a Borel set. Define

$$\mu_n := \frac{1}{n}|\{k : t_{n,k} \in B\}|, \qquad n = 1, 2, \ldots;$$

then

$$\lim_{n\to\infty} \mu_n = \mu$$

in the weak-star topology.

Proof. Let

$$w(x) = \left(1 + \frac{(1 + 2\lambda)}{\lambda^2} x^2\right)^{-\frac{1+\lambda}{2}}.$$

Let $t_n(x) = x^n + \cdots$ be the extremal polynomial for the sup norm on $[-1,1]$,

$$\|w(x)^n t_n(x)\| = \inf_{p_n = x^n + \cdots} \|w(x)^n p_n(x)\|,$$

and $T_n(x) = a_n^n t_n(\frac{x}{a_n})$. Notice that $a_n = \sqrt{(1+2\lambda)n/\lambda}$; then

$$\begin{aligned}
&\|w(x)^n H_n^{(N)}(a_n x)\|_{[-1,1],2} \\
&\quad = \left\| \left(1 + \frac{(1+2\lambda)}{\lambda^2} x^2\right)^{-\frac{1+\lambda}{2}n} H_n^{(N)}(a_n x)\right\|_{[-1,1],2} \\
&\quad = \left\| \left(1 + \frac{(1+2\lambda)}{\lambda^2} \frac{x^2}{a_n^2}\right)^{-\frac{1+\lambda}{2}n} H_n^{(N)}(x)\right\|_{[-a_n,a_n],2} \\
&\quad \le \left\| \left(1 + \frac{x^2}{\lambda n}\right)^{-\frac{1+\lambda}{2}n} H_n^{(N)}(x)\right\|_{(-\infty,\infty),2} \\
&\quad \le \left\| \left(1 + \frac{x^2}{\lambda n}\right)^{-\frac{1+\lambda}{2}n} T_n(x)\right\|_{(-\infty,\infty),2} \\
&\quad \le \left\| \left(1 + \frac{x^2}{\lambda n}\right)^{-\frac{1+\lambda}{2}n} T_n(x)\right\|_{-(\infty,\infty),\infty} \\
&\quad = \left\| \left(1 + \frac{x^2}{\lambda n}\right)^{-\frac{1+\lambda}{2}n} T_n(x)\right\|_{[-a_n,a_n],\infty} \\
&\quad = \left\| \left(1 + \frac{(1+2\lambda)}{\lambda^2} x^2\right)^{-\frac{1+\lambda}{2}n} T_n(a_n x)\right\|_{[-1,1],\infty} \\
&\quad = \left\| \left(1 + \frac{(1+2\lambda)}{\lambda^2} x^2\right)^{-\frac{1+\lambda}{2}n} a_n^n t_n(x)\right\|_{[-1,1],\infty}.
\end{aligned}$$

Thus we have

$$\left\| [w(x)]^n \frac{H_n(a_n x)}{a_n^n} \right\|_{[-1,1],2}^{\frac{1}{n}} \le \left\| \left(1 + \frac{(1+2\lambda)}{\lambda^2}\, x^2\right)^{-\frac{1+\lambda}{2}n} t_n(x)\right\|_{[-1,1],\infty}^{1/n}. \tag{5.1}$$

For the weight w, notice that $t_n(x)$ is the extremal for w, and from Lemma 1, we have

$$\limsup_{n\to\infty} \left\| w(x)^n \frac{H_n(a_n x)}{a_n^n} \right\|_{[-1,1],2}^{\frac{1}{n}} \le \exp(F_w).$$

Notice that $\frac{H_n(a_n x)}{a_n^n} = x^n + \cdots$ and the zeros of $H_n^{(N)}(a_n x)$ lie in $[-D, D]$. From Lemma 2, we have the proof of the theorem. □

Theorem 5.4. *Let $N = \lambda_n n$ and a_n be as in* (3.2). *If $\lambda_n \to \infty$, then*

$$\lim_{n\to\infty} |H_n^{(N)}(a_n x)|^{\frac{1}{n}} = \exp\left[\int_{-1}^{1} \log|z-t|\,d\nu(t)\right]$$

locally uniformly in $\mathbf{C}\backslash[-1,1]$, *where*

$$d\nu(t) = \frac{2}{\pi}\sqrt{1-t^2}\,dt.$$

Furthermore, let the sequence of unit measures $\{\nu_n\}_{n=1}^{\infty}$ *be*

$$\nu_n := \frac{1}{n}|\{k : t_{n,k} \in B\}|, \qquad n = 1, 2, \ldots,$$

where B *is a Borel set and* $\{t_{n,k}\}_{k=1}^{n}$ *are the zeros of* $H_n^{(N)}(a_n x)$. *If* $\lambda_n \to \infty$, *then*

$$\lim_{n-\infty} \nu_n = \nu$$

in the weak-star topology.

Proof. In the proof of (5.1), it is easy to see that for any $p_n(x) = x^n + \cdots \in \mathcal{P}_n$, we have

$$\left\| \left(1 + \frac{(1+2\lambda_n)}{\lambda_n^2}\, x^2\right)^{-\frac{1+\lambda_n}{2}n} \frac{H_n(a_n x)}{a_n^n} \right\|_{[-1,1],2} \tag{5.2}$$
$$\leq \left\| \left(1 + \frac{(1+2\lambda_n)}{\lambda_n^2}\, x^2\right)^{-\frac{1+\lambda_n}{2}n} p_n(x) \right\|_{[-1,1],\infty}.$$

Here, we consider $w(x) = e^{-x^2}$ on $[-1,1]$; the equilibrium measure is $d\nu = \frac{2}{\pi}\sqrt{1-t^2}\, dt$ [7]. Choose $p_n(x) = T_n(x; w, \infty)$; from (5.2), we have

$$\begin{aligned}
&\left\| [w(x)]^n \frac{H_n^{(N)}(a_n x)}{a_n^n} \right\|_{[-1,1],2} \\
&= \left\| e^{-nx^2} \frac{H_n^{(N)}(a_n x)}{a_n^n} \right\|_{[-1,1],2} \\
&\leq \left\| e^{-nx^2} \left(1 + \frac{(1+2\lambda_n)}{\lambda_n^2} x^2\right)^{\frac{1+\lambda_n}{2}n} \right\|_{[-1,1],2} \left\| \left(1 + \frac{(1+2\lambda_n)}{\lambda_n^2} x^2\right)^{-\frac{1+\lambda_n}{2}n} \frac{H_n^{(N)}(a_n x)}{a_n^n} \right\|_{[-1,1],2} \\
&\leq \left\| e^{-nx^2} \left(1 + \frac{(1+2\lambda_n)}{\lambda_n^2} x^2\right)^{\frac{1+\lambda_n}{2}n} \right\|_{[-1,1],2} \left\| \left(1 + \frac{(1+2\lambda_n)}{\lambda_n^2} x^2\right)^{-\frac{1+\lambda_n}{2}n} T_n(x; w, \infty) \right\|_{[-1,1],\infty} \\
&\leq \left\| e^{-nx^2} \left(1 + \frac{(1+2\lambda_n)}{\lambda_n^2} x^2\right)^{\frac{1+\lambda_n}{2}n} \right\|_{[-1,1],2} \left\| e^{nx^2} \left(1 + \frac{(1+2\lambda_n)}{\lambda_n^2} x^2\right)^{-\frac{1+\lambda_n}{2}n} \right\|_{[-1,1],\infty} \\
&\times \left\| e^{-nx^2} T_n(x; w, \infty) \right\|_{[-1,1],\infty}.
\end{aligned}$$

Notice that, as $\lambda_n \to \infty$,

$$\lim_{n\to\infty} \left\| e^{-nx^2} \left(1 + \frac{(1+2\lambda_n)}{\lambda_n^2} x^2\right)^{\frac{1+\lambda_n}{2}n} \right\|_{[-1,1],2}^{1/n} = 1$$

and

$$\lim_{n\to\infty} \left\| e^{nx^2} \left(1 + \frac{(1+2\lambda_n)}{\lambda_n^2} x^2\right)^{-\frac{1+\lambda_n}{2}n} \right\|_{[-1,1],\infty}^{1/n} = 1;$$

then

$$\lim_{n\to\infty} \left\| [w(x)]^n \frac{H_n^{(N)}(a_n x)}{a_n^n} \right\|_{[-1,1],2}^{1/n} \leq e^{F_w}.$$

From Lemma 2, this completes the proof of the theorem. □

For the case when N is fixed and $n \to \infty$, see [8].

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