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MAXIMAL CLIFFORD SEMIGROUPS OF MATRICES

EDMOND W. H. LEE

ABSTRACT. All maximal Clifford semigroups of matrices are identified up to isomorphism. If the ground field of the matrices is finite, then there exists a unique Clifford semigroup of maximum order.

1. INTRODUCTION

A *Clifford semigroup* is a regular semigroup with central idempotents. Clifford semigroups are precisely the completely regular semigroups that are also inverse semigroups. The reader is referred to [2] for more information on Clifford semigroups.

The set $M_n(\mathcal{F})$ of all $n \times n$ matrices over a field \mathcal{F} is a semigroup under usual matrix multiplication. In this article, Clifford semigroups in $M_n(\mathcal{F})$ are investigated. It is shown that up to isomorphism, the number of distinct maximal Clifford semigroups in $M_n(\mathcal{F})$ is precisely the number of partitions of the integer n . Furthermore, if \mathcal{F} is finite, then the semigroup $GL_n(\mathcal{F}) \cup \{\mathbf{0}\}$ (the general linear group with the $n \times n$ zero matrix $\mathbf{0}$ adjoined) is the unique Clifford semigroup in $M_n(\mathcal{F})$ of maximum order.

The reader is referred to [2] for all undefined notation and terminology of semigroup theory. A property of $M_n(\mathcal{F})$ that is important to the present investigation of semigroups in $M_n(\mathcal{F})$ is:

Lemma 1. *All semilattices in $M_n(\mathcal{F})$ are finite.*

Proof. This follows from well-known results. See [3, Lemma 2.1]. □

2. MAXIMAL CLIFFORD SEMIGROUPS IN $M_n(\mathcal{F})$

We first recall the construction and some properties of Σ -semigroups introduced in [3]. Basically, these semigroups are unions of sums of completely simple (multiplicative) semigroups in rings containing no infinite semilattices. However, for this article, it suffices to consider Σ -semigroups in the special case of unions of sums of groups in $M_n(\mathcal{F})$.

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Let $\mathcal{G} = \{G_i : i \in \Lambda\}$ be a collection of groups in $M_n(\mathcal{F})$ such that $G_i \neq \{\mathbf{0}\}$ for all $i \in \Lambda$. Suppose further that \mathcal{G} is *orthogonal*, that is, $G_i G_j = \{\mathbf{0}\}$ whenever $i \neq j$. Let e_i be the identity element of G_i . Since $\{\mathbf{0}\} \cup \{e_i : i \in \Lambda\}$ is a semilattice in $M_n(\mathcal{F})$, the set Λ (and hence \mathcal{G}) must be finite by Lemma 1.

For any subset I of Λ , define

$$G_I = \begin{cases} \{\mathbf{0}\} & \text{if } I = \emptyset \\ \sum_{i \in I} G_i & \text{otherwise.} \end{cases}$$

Lemma 2. *Let $S = \bigcup_{I \subseteq \Lambda} G_I$. Then:*

- (1) *The sets in $\{G_I : I \subseteq \Lambda\}$ are pairwise disjoint;*
- (2) *If $I \neq \emptyset$, then G_I is a group isomorphic to the direct product $\prod_{i \in I} G_i$;*
- (3) *S is a Clifford semigroup;*
- (4) *S is completely determined by the groups in \mathcal{G} .*

Proof. Parts (1), (2), and (3) follow from Lemma 3.1, Lemma 3.2, and Theorem 3.3 in [3], respectively. Part (4) follows easily from parts (1) and (2) and the definition of S . \square

The semigroup S in Lemma 2 is called the Σ -semigroup with foundation \mathcal{G} . More generally, by a Σ -semigroup we mean a semigroup S in $M_n(\mathcal{F})$ for which there exists an orthogonal collection \mathcal{G} of groups such that S is the Σ -semigroup with foundation \mathcal{G} .

We now identify all maximal Clifford semigroups in $M_n(\mathcal{F})$ up to isomorphism. Suppose S is a maximal Clifford semigroup in $M_n(\mathcal{F})$. Then S is a Σ -semigroup by [3, Corollary 6.2], whence we may assume $S = \bigcup_{I \subseteq \Lambda} G_I$ with foundation $\mathcal{G} = \{G_i : i \in \Lambda\}$. The idempotents of S commute so that they are simultaneously diagonalizable (by the matrix a , say). Since the conjugation map $x \mapsto axa^{-1}$ is an isomorphism, we may assume without loss of generality that the idempotents of S are diagonal matrices with entries from $\{0, 1\}$.

Since $G_i G_j = \{\mathbf{0}\}$ whenever $i \neq j$, the diagonal matrices e_i and e_j do not share any common nonzero diagonal entry. Suppose the (k, k) -entry of every e_i is 0. Let f denote the matrix in $M_n(\mathcal{F})$ with 1 in its (k, k) -entry and 0 everywhere else. Then f is an idempotent that does not belong to S . Moreover, since $f G_i = G_i f = \{\mathbf{0}\}$ for all $i \in \Lambda$ so that $f G_I = G_I f = \{\mathbf{0}\}$ for all $I \subseteq \Lambda$, the set $S \cup \{f\}$ is a Clifford semigroup that strictly contains S , contradicting the maximality of S . Consequently, f does not exist, whence for each $k \in \{1, \dots, n\}$, exactly one e_i has 1 in its (k, k) -entry. Equivalently, the sets

$$E_i = \{k : \text{the } (k, k)\text{-entry of } e_i \text{ is } 1\},$$

where $i \in \Lambda$, form a partition of the set $\{1, \dots, n\}$. Note that $|E_i| = \text{rank}(e_i)$. By conjugating S with an appropriate permutation matrix and relabelling of the indices i , we may assume that the integers in each E_i are consecutive, and that the integers $|E_i|$ ($i \in \Lambda$) are in non-increasing order. For example, consider the idempotent matrices

$$e_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

in $M_5(\mathcal{F})$. Then after conjugation by the permutation matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and relabelling of indices, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where $E_1 = \{1, 2\}$, $E_2 = \{3, 4\}$, $E_3 = \{5\}$, and $|E_1| \geq |E_2| \geq |E_3|$.

It remains to determine which groups each G_i in \mathcal{G} can possibly be, keeping in mind that \mathcal{G} must be orthogonal. Let $M_{E_i}(\mathcal{F})$ denote the set of all matrices in $M_n(\mathcal{F})$ with 0 in their (s, t) -entries for all $(s, t) \notin E_i \times E_i$. Since $G_i = e_i G_i e_i \subseteq M_{E_i}(\mathcal{F})$ and $M_{E_j}(\mathcal{F}) M_{E_k}(\mathcal{F}) = \{\mathbf{0}\}$ whenever $j \neq k$, the required property of \mathcal{G} being orthogonal will not be violated as long as G_i is chosen to be any group in $M_{E_i}(\mathcal{F})$ (with identity element e_i). But by the maximality of S , the group G_i must contain all matrices in $M_{E_i}(\mathcal{F})$ of rank $|E_i|$ (so that $G_i \cong \text{GL}_{|E_i|}(\mathcal{F})$).

We have thus shown:

Proposition 3. *Up to isomorphism, each maximal Clifford semigroup in $M_n(\mathcal{F})$ is a Σ -semigroup $\bigcup_{I \subseteq \Lambda} G_I$ with foundation $\mathcal{G} = \{G_i : i \in \Lambda\}$, and there exists a partition $\{E_i : i \in \Lambda\}$ of the set $\{1, \dots, n\}$ such that $G_i \cong \text{GL}_{|E_i|}(\mathcal{F})$ for all $i \in \Lambda$. Consequently, the number of non-isomorphic maximal Clifford semigroups in $M_n(\mathcal{F})$ is precisely the number of partitions of n .*

Let $P = (n_1, \dots, n_r)$ be a partition of the integer n , that is, n_1, \dots, n_r are positive integers in non-increasing order such that $n_1 + \dots + n_r = n$. In view of Proposition 3, up to isomorphism, P corresponds uniquely to the maximal Clifford semigroup $\bigcup\{G_I : I \subseteq \{1, \dots, r\}\}$ with foundation $\mathcal{G} = \{G_1, \dots, G_r\}$, where $G_i \cong \text{GL}_{n_i}(\mathcal{F})$. This maximal Clifford semigroup

is said to be *associated with* the partition P and is denoted by C^P . Note then that $G_I \cong \prod_{i \in I} \text{GL}_{n_i}(\mathcal{F})$.

3. THE CLIFFORD SEMIGROUP IN $M_n(\mathcal{F})$ OF MAXIMUM ORDER

In this section, we assume \mathcal{F} is a finite field with q elements. Since $M_n(\mathcal{F})$ is already a Clifford semigroup (of maximum order q) when $n = 1$, we may also assume that $n \geq 2$, whence the order of $\text{GL}_n(\mathcal{F})$ is

$$\gamma(n) = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}) = \prod_{i=0}^{n-1} (q^n - q^i)$$

(see, for example, [1]). For any partition $P = (n_1, \dots, n_r)$ of n , define

$$\begin{aligned} \sigma_0(n, P) &= 1, \\ \sigma_1(n, P) &= \gamma(n_1) + \gamma(n_2) + \cdots + \gamma(n_r), \\ \sigma_2(n, P) &= \gamma(n_1)\gamma(n_2) + \gamma(n_1)\gamma(n_3) + \cdots + \gamma(n_{r-1})\gamma(n_r), \\ &\vdots \\ \sigma_k(n, P) &= \sum \{ \gamma(n_{i_1}) \cdots \gamma(n_{i_k}) : 1 \leq i_1 < \cdots < i_k \leq r \}, \\ &\vdots \\ \sigma_r(n, P) &= \gamma(n_1) \cdots \gamma(n_r). \end{aligned}$$

By Lemma 2(2), the order of $\text{GL}_{n_{i_1}}(\mathcal{F}) \times \cdots \times \text{GL}_{n_{i_k}}(\mathcal{F})$ is $\gamma(n_{i_1}) \cdots \gamma(n_{i_k})$. Therefore $\sigma_k(n, P)$ is the sum of all $|G_I|$ where $|I| = k$. Hence:

Theorem 4. *The order of the maximal Clifford semigroup C^P in $M_n(\mathcal{F})$ is*

$$\sigma(n, P) = \sigma_0(n, P) + \cdots + \sigma_r(n, P) = \sum_{k=0}^r \sigma_k(n, P).$$

Note that if T is the trivial partition (n) of n , then $C^T = \text{GL}_n(\mathcal{F}) \cup \{\mathbf{0}\}$ and $\sigma(n, T) = 1 + \gamma(n)$. For the rest of this article, we show that C^T is the unique Clifford semigroup in $M_n(\mathcal{F})$ of maximum order.

Lemma 5. *Suppose $n = s + t$ where $s, t \geq 1$. Then $4\gamma(s)\gamma(t) \leq \gamma(n)$.*

Proof. Since

$$\begin{aligned} 2\gamma(s) &= 2(q^s - 1)(q^s - q) \cdots (q^s - q^{s-1}) \\ &\leq (q^n - 1)(q^n - q) \cdots (q^n - q^{s-1}) \end{aligned}$$

and

$$\begin{aligned} 2\gamma(t) &= 2(q^t - 1)(q^t - q) \cdots (q^t - q^{t-1}) \\ &= 2 \cdot \frac{q^{s+t} - q^s}{q^s} \cdot \frac{q^{s+t} - q^{s+1}}{q^s} \cdots \frac{q^{s+t} - q^{s+t-1}}{q^s} \\ &\leq (q^n - q^s)(q^n - q^{s+1}) \cdots (q^n - q^{n-1}), \end{aligned}$$

we have $4\gamma(s)\gamma(t) \leq \prod_{i=0}^{s-1}(q^n - q^i)\prod_{j=s}^{n-1}(q^n - q^j) = \gamma(n)$. \square

Lemma 6. *If P is any partition of n , then $\sigma(n, P) \leq 1 + \gamma(n)$.*

Proof. It suffices to assume that P is nontrivial. We proceed by induction on n . For $n = 2$, the only nontrivial partition is $P = (1, 1)$, whence

$$\begin{aligned}\sigma(2, P) &= 1 + 2(q - 1) + (q - 1)^2 \\ &\leq 1 + (q^2 - 1)(q^2 - q) \\ &= 1 + \gamma(2)\end{aligned}$$

for all $q \geq 2$. Suppose the inequality holds for all integers strictly less than n . Let $P = (n_1, \dots, n_r)$ be a nontrivial partition of n . Note that for $1 \leq k \leq r$,

$$\begin{aligned}\sigma_k(n, P) &= \sum\{\gamma(n_{i_1}) \cdots \gamma(n_{i_k}) : 1 \leq i_1 < \cdots < i_k \leq r\} \\ &= \sum\{\gamma(n_{i_1}) \cdots \gamma(n_{i_k}) : 1 \leq i_1 < \cdots < i_k \leq r - 1\} \\ &\quad + \sum\{\gamma(n_{i_1}) \cdots \gamma(n_{i_{k-1}}) \gamma(n_r) : 1 \leq i_1 < \cdots < i_{k-1} \leq r - 1\} \\ &= \sigma_k(n - n_r, P') + \sigma_{k-1}(n - n_r, P') \cdot \gamma(n_r),\end{aligned}$$

where P' is the partition (n_1, \dots, n_{r-1}) of $n - n_r$. Hence

$$\begin{aligned}\sigma(n, P) &= 1 + \sum_{k=1}^r \sigma_k(n, P) \\ &= (1 + \sum_{k=1}^{r-1} \sigma_k(n - n_r, P')) + \sigma_r(n - n_r, P') \\ &\quad + \gamma(n_r) \sum_{k=1}^{r-1} \sigma_{k-1}(n - n_r, P') \\ &= \sigma(n - n_r, P') + 0 + \gamma(n_r) \cdot \sigma(n - n_r, P') \\ &= (1 + \gamma(n_r)) \cdot \sigma(n - n_r, P').\end{aligned}$$

Since $\sigma(n - n_r, P') \leq 1 + \gamma(n - n_r)$ by induction hypothesis, we have

$$\begin{aligned}\sigma(n, P) &\leq (1 + \gamma(n_r)) \cdot (1 + \gamma(n - n_r)) \\ &\leq (2\gamma(n_r)) \cdot (2\gamma(n - n_r)) \\ &\leq 1 + \gamma(n),\end{aligned}$$

where the last inequality holds by Lemma 5. \square

Theorem 7. *Let \mathcal{F} be a finite field. Then $\text{GL}_n(\mathcal{F}) \cup \{\mathbf{0}\}$ is the unique Clifford semigroup in $\text{M}_n(\mathcal{F})$ of maximum order.*

Proof. By Proposition 3 and Lemma 6, a Clifford semigroup S in $\text{M}_n(\mathcal{F})$ of maximum order is isomorphic to $C^T = \text{GL}_n(\mathcal{F}) \cup \{\mathbf{0}\}$. Since $\text{GL}_n(\mathcal{F})$ is the unique maximal group in $\text{M}_n(\mathcal{F})$ of rank n , we have $S = \text{GL}_n(\mathcal{F}) \cup \{\mathbf{0}\}$. \square

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