

Harnack Inequalities: from Harmonic Analysis through Poincare Conjecture to Matrix Determinant

Fuzhen Zhang

zhang@nova.edu

Joint work with Dr. Minghua Lin

Who is Harnack?

Carl Gustav Axel Harnack (1851-1888),
born in Tartu, Estonia, died in Dresden, Germany.
PhD in 1875 from Felix Klein (..... Klein bottle



wikipedia.org

The classical Harnack inequality in Potential Theory

Axel Harnack's book, pp.158, 1887, *Die Grundlagen der Theorie des logarithmischen Potentials und der eindeutigen Potentialfunktion in der Ebene*, Leipzig: V. G. Teubner

In English: *Foundations of the theory of the logarithmic potential and single-valued potential functions in the plane*

in which an inequality of a positive harmonic function was introduced, later generalized to solutions of elliptic or parabolic partial differential equations. Perelman's solution (2003) of the Poincaré conjecture uses a version of the Harnack inequality, found by R. Hamilton (1993), for the Ricci flow.

The Harnack inequality in Functional Analysis

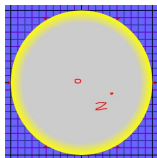
Let $f(z)$ be a positive harmonic function on $|z| < 1$ in the plane.

Then

$$f(0) \frac{1 - |z|}{1 + |z|} \leq f(z) \leq f(0) \frac{1 + |z|}{1 - |z|}$$

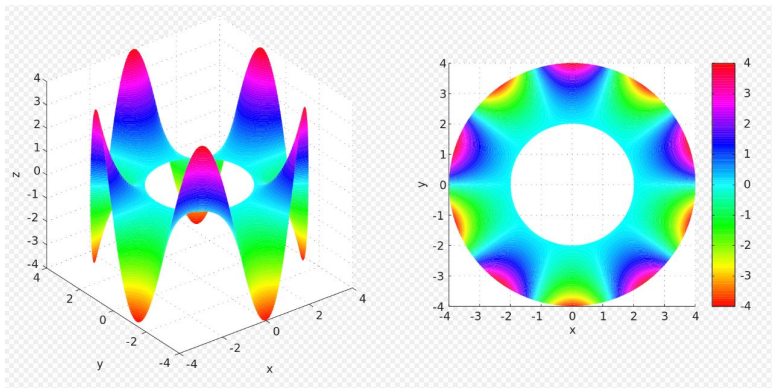
Recall that a harmonic function is a twice continuously differentiable function on an open set (in \mathbb{R}^n or \mathbb{C}^n) satisfying the Laplace equation

$$\nabla^2 f = 0$$



Example: Take a point z on $|z| = 0.5$. Then $\frac{1}{3}f(0) \leq f(z) \leq 3f(0)$.

A harmonic function defined on an annulus



Courtesy: en.wikipedia.org

The Harnack inequality in Functional Analysis

Set $z = re^{i\theta}$, where r is the modulus, θ is the argument of z . Then

$$f(0) \frac{1-r}{1+r} \leq f(z) \leq f(0) \frac{1+r}{1-r}$$

If we scale and translate to an arbitrary disk of radius R with center z_0 , then we have for $f(z)$, a positive harmonic function on $|z| < R$,

$$f(z_0) \frac{R-r}{R+r} \leq f(z) \leq f(z_0) \frac{R+r}{R-r}, \quad |z - z_0| < r < R$$

The Harnack inequality in higher dimension

Denote the open ball (in usual topology) centered at x_0 with radius R in the n -dimensional space \mathbb{R}^n by

$$B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$$

Consider

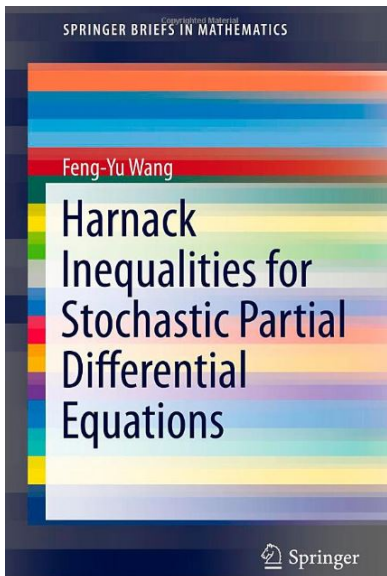
$$B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r < R\}$$

Then for any z on the surface of $B_r(x_0)$, i.e., $|z - x_0| = r$, we have

$$f(z_0) \frac{1 - \rho}{(1 + \rho)^{n-1}} \leq f(z) \leq f(z_0) \frac{1 + \rho}{(1 - \rho)^{n-1}}, \quad \rho = \frac{r}{R}$$

(Extensions for general domains; proof by Poisson's formula \int_{sphere})

The Harnack-type inequalities in PDEs



When Pf is measurable for $f \in \mathcal{B}_b(E)$, i.e., the family $\{\mu_t : x \in E\}$ is a transition probability measure, then P is a Markov operator on $\mathcal{B}_b(E)$, i.e., P is contractive and positivity-preserving, and $P1 = 1$.

When a family of stochastic processes $\{(X^t(t))_{t \geq 0} : x \in E\}$ measurable in x is involved, for instance $(X^t(t))_{t \geq 0}$ solves a stochastic differential equation with initial data x , let $\mu_t(t)$ be the distribution of $X^t(t)$. Then we define as in (1.1) a family of Markov operators $(P_t)_{t \geq 0}$. If the family of processes is Markovian, then $(P_t)_{t \geq 0}$ is a semigroup, i.e., $P_{t+s} = P_t P_s$ for $t, s \geq 0$.

In the remainder of the section, we will use coupling by change of measure to establish Harnack-type inequalities and derivative formulas of P .

1.1.1 Harnack Inequalities and Bismut Derivative Formulas

For a Markov operator P on $\mathcal{B}_b(E)$, the Harnack-type inequality considered in this book is of type

$$\Phi(Pf(x)) \leq (P\Phi(f)(y))e^{\Psi(x,y)}, \quad x, y \in E, f \in \mathcal{B}_b^+(E), \quad (1.2)$$

where Φ is a nonnegative convex function on $[0, \infty)$ and Ψ is a nonnegative function on E^2 . By Jensen's inequality, we may always take $\Psi(x, x) = 0$.

In this book we will mainly consider the following two typical choices of Φ :

- (1) **(Harnack inequality with power)** Let $\Phi(r) = r^p$ for some $p > 1$. Then (1.2) reduces to

$$(Pf(x))^p \leq (Pf^p(y))e^{\Psi(x,y)}, \quad x, y \in E, f \in \mathcal{B}_b^+(E). \quad (1.3)$$

This inequality, called the Harnack inequality with power p , was first found in [50] for diffusion semigroups with curvature bounded from below.

- (2) **(Log-Harnack inequality)** Let $\Phi(r) = e^r$. In this case we may use $\log f$ to replace f , so that (1.2) is equivalent to

$$P \log f(x) \leq \log Pf(y) + \Psi(x, y), \quad x, y \in E, f \in \mathcal{B}_b^+(E), f \geq 1. \quad (1.4)$$

Since the inequality does not change by multiplying f by a positive constant, it holds for all uniformly positive functions f . Moreover, using $f + \varepsilon$ to replace f and letting $\varepsilon \downarrow 0$, we may replace the condition $f \geq 1$ by $f \geq 0$. This inequality, called the log-Harnack inequality, was introduced in [56] for reflecting diffusion processes on manifolds with boundary and in [43] for semilinear SPDEs (stochastic partial differential equations) with multiplicative noise.

Theorem 1.1.1 (Harnack inequalities) *If there is a coupling by change of measure (X, Y) for μ_x and μ_y , with changed probability $d\mathbb{Q} := Rd\mathbb{P}$, such that $X = Y, \mathbb{Q} \ll \mathbb{P}$, then*

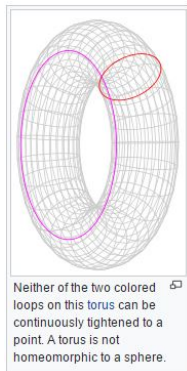
Poincaré Conjecture: A \$1M Millennium Prize Problem

Poincaré conjecture (1904-2003):

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.



Courtesy: en.wikipedia.org



Perelman's Harnack inequality in his solution to Poincaré Conjecture

Perelman resolved the Poincaré conjecture in 2003...

Perelman's solution uses a version of the Harnack inequality for the Ricci flow, found by R. Hamilton (1993), which is an extension of a result of P. Li and S.-T. Yau (1986).

A MATRIX HARNACK ESTIMATE FOR THE HEAT EQUATION

RICHARD S. HAMILTON

In the important paper [LY] by Peter Li and S.-T. Yau, they show how the classical Harnack principle for the heat equation on a manifold can be derived from a differential inequality. In particular, they show that for any positive solution $f > 0$ of the heat equation

$$\frac{\partial f}{\partial t} = \Delta f$$

on a compact Riemannian manifold of dimension m solving the equation for $t > 0$, if the manifold has weakly positive Ricci curvature $R_{ij} \geq 0$ then for any vector field V on M

$$\frac{\partial f}{\partial t} + \frac{1}{2t} f + 2Df(V) + f|V|^2 \geq 0;$$

a similar result holds with an error term if the Ricci curvature is bounded below. The quadratic version in V given here is equivalent to the more complicated formula in their paper by choosing the optimal V . We shall show in this paper that the Harnack estimate of Li and Yau is the trace of a full matrix inequality.

Main Theorem. *If M is a compact Riemannian manifold and $f > 0$ is a positive solution to the heat equation on M*

$$\frac{\partial f}{\partial t} = \Delta f$$

for $t > 0$, then for any vector field V_i on M we have

$$D_i D_j f + \frac{1}{2t} f g_{ij} + D_j f \cdot V_j + D_j f \cdot V_i + f V_j V_j \geq 0$$

Research partially supported by NSF contract DMS 90-03333.

9 Differential Harnack inequality for solutions of the conjugate heat equation

9.1 Proposition. *Let $g_{ij}(t)$ be a solution to the Ricci flow $(g_{ij})_t = -2R_{ij}$, $0 \leq t \leq T$, and let $u = (4\pi(T-t))^{-\frac{n}{2}} e^{-f}$ satisfy the conjugate heat equation $\square^* u = -u_t - \Delta u + Ru = 0$. Then $v = [(T-t)(2\Delta f - |\nabla f|^2 + R) + f - n]$ satisfies*

$$\square^* v = -2(T-t)|R_{ij} + \nabla_i \nabla_j f - \frac{1}{2(T-t)} g_{ij}|^2 \quad (9.1)$$

Proof. Routine computation.

Clearly, this proposition immediately implies the monotonicity formula (3.4); its advantage over (3.4) shows up when one has to work locally.

9.2 Corollary. *Under the same assumptions, on a closed manifold M , or whenever the application of the maximum principle can be justified, $\min v/u$ is nondecreasing in t .*

9.3 Corollary. *Under the same assumptions, if u tends to a δ -function as $t \rightarrow T$, then $v \leq 0$ for all $t < T$.*

Proof. If h satisfies the ordinary heat equation $h_t = \Delta h$ with respect to the evolving metric $g_{ij}(t)$, then we have $\frac{d}{dt} \int h u = 0$ and $\frac{d}{dt} \int h v \geq 0$. Thus we only need to check that for everywhere positive h the limit of $\int h v$ as $t \rightarrow T$ is nonpositive. But it is easy to see, that this limit is in fact zero.

9.4 Corollary. *Under assumptions of the previous corollary, for any smooth curve $\gamma(t)$ in M holds*

$$-\frac{d}{dt} f(\gamma(t), t) \leq \frac{1}{2}(R(\gamma(t), t) + |\dot{\gamma}(t)|^2) - \frac{1}{2(T-t)} f(\gamma(t), t) \quad (9.2)$$

Proof. From the evolution equation $f_t = -\Delta f + |\nabla f|^2 - R + \frac{n}{2(T-t)}$ and $v \leq 0$ we get $f_t + \frac{1}{2} R - \frac{1}{2} |\nabla f|^2 - \frac{1}{2(T-t)} f \geq 0$. On the other hand, $-\frac{d}{dt} f(\gamma(t), t) = -f_t - \langle \nabla f, \dot{\gamma}(t) \rangle \geq -f_t + \frac{1}{2} |\nabla f|^2 + \frac{1}{2} |\dot{\gamma}|^2$. Summing these two inequalities, we get (9.2).

Ky Fan's Harnack type inequality for Operators

Theorem (Fan 1988)

Let F be an operator-valued analytic function on the open unit disk $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that for any $z \in \mathfrak{D}$, $F(z)$ is an operator on a complex Hilbert space \mathbb{H} with $\operatorname{Re} F(z) > 0$ and $F(0) = I$. Then

$$\frac{1 - |z|}{1 + |z|} I \leq \operatorname{Re} F(z) \leq \frac{1 + |z|}{1 - |z|} I$$

Proof. For each x in \mathbb{H} with $\|x\| = 1$, define the complex-valued $f_x(z) = \langle F(z)x, x \rangle$. Use the classical Harnack inequality. \square

Note: There is an analog for $\operatorname{Im} F(z)$.

Tung's Harnack type inequality for matrices

Theorem (Tung 1964)

Let Z be an $n \times n$ complex matrix with singular values r_k that satisfy $0 \leq r_k < 1$, $k = 1, 2, \dots, n$ (i.e., Z is a strict contraction). Let Z^* denote the conjugate transpose of Z and I be the $n \times n$ identity matrix. Then for any $n \times n$ unitary matrix U

$$\prod_{k=1}^n \frac{1 - r_k}{1 + r_k} \leq \frac{\det(I - Z^*Z)}{|\det(I - UZ)|^2} \leq \prod_{k=1}^n \frac{1 + r_k}{1 - r_k} \quad (1)$$

Proof. Consider $f(U) = \det((I - ZU^*)(I - UZ^*))$ for fixed strict contraction Z and use the method of Lagrange multipliers. \square

Marcus (1965) gave another proof and pointed out that Tung's inequality is equivalent to

$$\prod_{k=1}^n (1 - r_k) \leq |\det(I - A)| \leq \prod_{k=1}^n (1 + r_k) \quad (2)$$

for any $n \times n$ matrix A with the same singular values as the contractive matrix Z .

L.-K. Hua (1965) gave a proof of (2) using an inequality he had previously obtained in 1955: For strict contractions A, B ,

$$\begin{pmatrix} (I - A^*A)^{-1} & (I - B^*A)^{-1} \\ (I - A^*B)^{-1} & (I - B^*B)^{-1} \end{pmatrix} \geq 0$$

- (i). In the book by Marshall, Olkin and Arnold, Tung's theorem is cited in which the condition that A be contractive is missing.
- (ii). Inequalities (1) and (2) are not equivalent for general matrices. The right-hand side inequality in (2) is true for all $n \times n$ matrices A ; that is,

$$|\det(I - A)| \leq \prod_{k=1}^n (1 + r_k)$$

Restatement of Tung's Theorem and equality case

Theorem (Left inequality)

Let Z be an $n \times n$ positive semidefinite matrix with eigenvalues r_1, r_2, \dots, r_n . Let U be an $n \times n$ unitary matrix such that $I - UZ$ is nonsingular. Then

$$\prod_{k=1}^n \frac{|1 - r_k|}{1 + r_k} \leq \frac{|\det(I - Z^2)|}{|\det(I - UZ)|^2} \quad (3)$$

with equality if and only if Z has an eigenvalue 1 or UZ has eigenvalues $-r_1, -r_2, \dots, -r_n$. If both Z and $I - Z$ are nonsingular, the strict inequality holds for $U \neq -I$.

Restatement of Tung's Theorem and equality case

Theorem (Right inequality)

Let Z be an $n \times n$ positive semidefinite matrix with eigenvalues r_1, r_2, \dots, r_n . Let U be an $n \times n$ unitary matrix such that $I - UZ$ is nonsingular. If $0 \leq r_k < 1$, $k = 1, 2, \dots, n$, then

$$\frac{\det(I - Z^2)}{|\det(I - UZ)|^2} \leq \prod_{k=1}^n \frac{1 + r_k}{1 - r_k} \quad (4)$$

with equality if and only if UZ has eigenvalues r_1, r_2, \dots, r_n . If Z is nonsingular, then the strict inequality in (4) holds if $U \neq I$.

Majorization

Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be real vectors whose components are arranged in nonincreasing order:

$$x_1 \geq x_2 \geq \dots \geq x_n, \quad y_1 \geq y_2 \geq \dots \geq y_n$$

If

$$\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i, \quad k = 1, 2, \dots, n$$

we say that x is weakly majorized by y , written $x \prec_w y$. If the last inequality becomes equality, then x is majorized by y , denoted $x \prec y$.

$$x \prec_w y, \quad x \prec y$$

Replacing \sum by \prod , we have log-majorization:

$$x \prec_{w \log} y, \quad x \prec_{\log} y$$

Lemma

Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be nonnegative vectors and assume that y is not a permutation of x (i.e., the multisets $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$ are not equal). Denote $\tilde{z} = (1 + z_1, 1 + z_2, \dots, 1 + z_n)$. We have:

$$\text{If } x \prec_{\log} y, \quad \text{then } \tilde{x} \prec_{w\log} \tilde{y}$$

and

$$\prod_{k=1}^n (1 + x_k) < \prod_{k=1}^n (1 + y_k). \quad (5)$$

Proof. $f(t) = \ln(1 + e^t)$ is strictly increasing & convex on $(0, \infty)$. \square

Lemma

If all $x_i, y_i \in [0, 1)$, x is not a permutation of y , and $x \prec_{\log} y$, then

$$\prod_{k=1}^n (1 - x_k) > \prod_{k=1}^n (1 - y_k). \quad (6)$$

Proof. $-\ln(1 - e^t)$ is strictly increasing and convex on $(-\infty, 0)$. \square

Proof of the equality case of the theorems

Proof of the Theorems. Use Majorization Theory.

Only show the equality cases. For (3), if Z has a singular (eigen-) value 1, then both sides vanish. If UZ has eigenvalues $-r_1, -r_2, \dots, -r_n$, then $\det(I - UZ) = \prod_{k=1}^n (1 + r_k)$. Equality is readily seen. Conversely, suppose equality occurs in (3). We further assume that no r_k ($k = 1, 2, \dots, n$) equals 1. Since $|\det(I - Z^2)| = \prod_{k=1}^n |1 - r_k|(1 + r_k)$, we have

$$|\det(I - UZ)| = \prod_{k=1}^n (1 + r_k). \quad (7)$$

Moreover, by **Weyl majorization inequality**

$$|\lambda(UZ)| \prec_{\log} \sigma(UZ) = \sigma(Z) = \lambda(Z),$$

where $\lambda(X)$ and $\sigma(X)$ denote the vectors of the eigenvalues and singular values of matrix X , respectively.

With $\lambda_k(X)$ denoting the eigenvalues of the $n \times n$ matrix X , $k = 1, 2, \dots, n$, by the lemma, we have

$$0 < |\det(I - UZ)| = \prod_{k=1}^n |1 - \lambda_k(UZ)| \leq \prod_{k=1}^n (1 + |\lambda_k(UZ)|) \leq \prod_{k=1}^n (1 + r_k).$$

Thus, (7) yields $|1 - \lambda_k(UZ)| = 1 + |\lambda_k(UZ)|$ for all k , which implies $\lambda_k(UZ) \leq 0$ for $k = 1, 2, \dots, n$, i.e., all eigenvalues of $-UZ$ are nonnegative. If $|\lambda(UZ)| = \lambda(-UZ)$ is not a permutation of $\lambda(Z)$, then, by strict inequality (5), we have

$\prod_{k=1}^n (1 + |\lambda_k(UZ)|) < \prod_{k=1}^n (1 + \lambda_k(Z)) = \prod_{k=1}^n (1 + r_k)$, a contradiction to (7). It follows that UZ has the eigenvalues $-r_1, -r_2, \dots, -r_n$.

For the equality in (4), it occurs if and only if

$\prod_{k=1}^n (1 - r_k) = |\det(I - UZ)|$. Note that $|\lambda(UZ)| \prec_{\log} \sigma(Z)$ and

$$\prod_{k=1}^n |1 - \lambda_k(UZ)| \geq \prod_{k=1}^n (1 - |\lambda_k(UZ)|) \geq \prod_{k=1}^n (1 - \sigma_k(Z)) = \prod_{k=1}^n (1 - r_k). \quad (8)$$

The first equality in (8) occurs if and only if all $\lambda_k(UZ)$ are in $[0, 1]$; the second equality occurs if and only if $\lambda(UZ)$ is a permutation of $\sigma(Z)$, i.e., $\text{Spec}(UZ) = \text{Spec}(Z)$.

Now assume that Z is nonsingular and suppose that equality holds in (4). Then UZ has eigenvalues r_1, r_2, \dots, r_n . Moreover, the singular values of UZ are r_1, r_2, \dots, r_n . Let $P = UZ$. Then the eigenvalues of P are just the singular values of P . So P is positive definite. It follows that $U = PZ^{-1}$ has only positive eigenvalues. Since U is unitary, U has to be the identity matrix. The case for (3) is similar. \square

Extension of Tung's Theorem on Harnack inequality

Theorem (Lin and Z. 2017)

Let Z_i , $i = 1, 2, \dots, m$, be $n \times n$ positive semidefinite matrices. Suppose that the eigenvalues of Z_i are r_{ik} satisfying $0 \leq r_{ik} < 1$, $k = 1, 2, \dots, n$. Then for any $n \times n$ unitary matrix U and positive scalars w_i , $i = 1, 2, \dots, m$, $\sum_{i=1}^m w_i = 1$, we have

$$\prod_{k=1}^n \prod_{i=1}^m \left(\frac{1 - r_{ik}}{1 + r_{ik}} \right)^{w_i} \leq \frac{\det(I - (\sum_{i=1}^m w_i Z_i)^2)}{|\det(I - U \sum_{i=1}^m w_i Z_i)|^2} \leq \prod_{k=1}^n \prod_{i=1}^m \left(\frac{1 + r_{ik}}{1 - r_{ik}} \right)^{w_i}. \quad (9)$$

Equality on the left-hand side occurs if and only if all Z_i are equal to Z , say, and Z has an eigenvalue 1 or $\text{Spec}(UZ) = \text{Spec}(-Z)$ (in which $U = -I$ if Z is nonsingular); Equality on the right-hand side occurs if and only if all Z_i are equal to Z , say, and $\text{Spec}(UZ) = \text{Spec}(Z)$ (in which $U = I$ if Z is nonsingular).

Outline of the Proof

Fact: for $n \times n$ Hermitian A and B , if $\lambda_k(A + B) = \lambda_k(A) + \lambda_k(B)$ for all k , then A and B are simultaneously unitarily diagonalizable with their eigenvalues on the main diagonal in the same order.

Fan's majorization $\lambda(H + S) \prec \lambda(H) + \lambda(S)$ for $n \times n$ Hermitian matrices H and S and **Lewent's inequality** for $x_i \in [0, 1)$,

$$\frac{1 + \sum_{i=1}^n \alpha_i x_i}{1 - \sum_{i=1}^n \alpha_i x_i} \leq \prod_{i=1}^n \left(\frac{1 + x_i}{1 - x_i} \right)^{\alpha_i},$$

where $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i > 0$. Equality holds iff $x_1 = x_2 = \dots = x_n$.

Let r_{ik}^\downarrow be the k th largest eigenvalue of Z_i , $i = 1, 2, \dots, m$, and s_k be the k th largest eigenvalue of $W := \sum_{i=1}^m w_i Z_i$, $k = 1, 2, \dots, n$.

$$\lambda(W) \prec \sum_{i=1}^m w_i \lambda(Z_i), \quad \text{i.e.,} \quad \sum_{k=1}^{\ell} s_k \leq \sum_{k=1}^{\ell} \sum_{i=1}^m w_i r_{ik}^\downarrow, \quad \ell = 1, 2, \dots, n.$$

(Note that the components of $\lambda(\cdot)$ are in nonincreasing order.)

Now the convexity and the monotonicity of the function

$f(t) = \ln \frac{1+t}{1-t}$, $0 \leq t < 1$, imply

$$\sum_{k=1}^n \ln \frac{1+s_k}{1-s_k} \leq \sum_{k=1}^n \ln \frac{1+\sum_{i=1}^m w_i r_{ik}^\downarrow}{1-\sum_{i=1}^m w_i r_{ik}^\downarrow},$$

where equality holds if and only if $s_k = \sum_{i=1}^m w_i r_{ik}^\downarrow$ for all k ; that is, $\lambda(W) = \sum_{i=1}^m w_i \lambda(Z_i)$. It follows that all Z_i are simultaneously unitarily diagonalizable with their eigenvalues on the main diagonals in the same order (nonincreasing, say).

Applying the exponential function to both sides and using Lewent's inequality yield

$$\begin{aligned} \prod_{k=1}^n \frac{1+s_k}{1-s_k} &\leq \prod_{k=1}^n \frac{1+\sum_{i=1}^m w_i r_{ik}^\downarrow}{1-\sum_{i=1}^m w_i r_{ik}^\downarrow} \\ &\leq \prod_{k=1}^n \prod_{i=1}^m \left(\frac{1+r_{ik}^\downarrow}{1-r_{ik}^\downarrow} \right)^{w_i} \\ &= \prod_{k=1}^n \prod_{i=1}^m \left(\frac{1+r_{ik}}{1-r_{ik}} \right)^{w_i}, \end{aligned} \tag{10}$$

in which equality occurs in the second inequality if and only if $r_{1k} = r_{2k} = \cdots = r_{mk}$ for $k = 1, 2, \dots, n$. Thus both equalities in (10) hold if and only if $Z_1 = Z_2 = \cdots = Z_m$.

By (4), we have

$$\frac{\det(I - (\sum_{i=1}^m w_i Z_i)^2)}{|\det(I - U \sum_{i=1}^m w_i Z_i)|^2} \leq \prod_{k=1}^n \left(\frac{1 + s_k}{1 - s_k} \right). \quad (11)$$

Combining (10) and (11) gives the second inequality of (9).

Note that the inequalities in (10) reverse by taking reciprocals, which implies

$$\prod_{k=1}^n \frac{1 - s_k}{1 + s_k} \geq \prod_{k=1}^n \prod_{i=1}^m \left(\frac{1 - r_{ik}}{1 + r_{ik}} \right)^{w_i}. \quad (12)$$

Then by (3), we have

$$\frac{\det(I - (\sum_{i=1}^m w_i Z_i)^2)}{|\det(I - U \sum_{i=1}^m w_i Z_i)|^2} \geq \prod_{k=1}^n \left(\frac{1 - s_k}{1 + s_k} \right). \quad (13)$$

Combining (12) and (13) yields the first inequality of (9).

If either equality holds in (9), then all Z_i are equal to Z , say. The conclusions are immediate from Theorem 3. \square

Corollary

Let Z_i , $i = 1, 2, \dots, m$, be $n \times n$ complex matrices with singular values r_{ik} such that $0 \leq r_{ik} < 1$, $k = 1, 2, \dots, n$. Then for any $n \times n$ unitary matrix U

$$\frac{\det(I - \sum_{i=1}^m w_i Z_i^* Z_i)}{|\det(I - U \sum_{i=1}^m w_i |Z_i|)|^2} \leq \prod_{k=1}^n \prod_{i=1}^m \left(\frac{1 + r_{ik}}{1 - r_{ik}} \right)^{w_i},$$

where $w_i > 0$, $i = 1, 2, \dots, m$, such that $\sum_{i=1}^m w_i = 1$. Equality occurs if and only if all Z_i have the same absolute value, say Z , and $\text{Spec}(UZ) = \text{Spec}(Z)$ (in which $U = I$ if Z is nonsingular).

Proof. With $(\sum_{i=1}^m w_i |Z_i|)^2 \leq \sum_{i=1}^m w_i |Z_i|^2$. □

An example

In view of the inequality in the corollary, it is tempting to have the lower bound inequality

$$\prod_{k=1}^n \prod_{i=1}^m \left(\frac{1 - r_{ik}}{1 + r_{ik}} \right)^{w_i} \leq \frac{\det(I - \sum_{i=1}^m w_i Z_i^* Z_i)}{|\det(I - U \sum_{i=1}^m w_i |Z_i|)|^2}.$$

However, this is not true. Set $m = n = 2$, $w_1 = w_2 = 1/2$ and take

$$Z_1 = \begin{pmatrix} 0.34 & -0.15 \\ -0.15 & 0.07 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 0.02 & -0.01 \\ -0.01 & 0.01 \end{pmatrix},$$

$$U = \begin{pmatrix} -0.60 & 0.80 \\ 0.80 & 0.60 \end{pmatrix}.$$

One may check that the left hand side is 0.6281, while the right hand side is 0.6250.

A question

Replace Z with U^*Z and leave the singular values unchanged in the Theorem. Giving an upper bound and lower bound, in terms of the singular values of individual matrices, for the quantity

$\frac{\det(I - \sum_{i=1}^m w_i Z_i^* Z_i)}{|\det(I - \sum_{i=1}^m w_i Z_i)|^2}$, where Z_i , $i = 1, 2, \dots, m$, are general contractive matrices. We would guess

$$\prod_{k=1}^n \prod_{i=1}^m \left(\frac{1 - r_{ik}}{1 + r_{ik}} \right)^{w_i} \leq \frac{\det(I - \sum_{i=1}^m w_i Z_i^* Z_i)}{|\det(I - \sum_{i=1}^m w_i Z_i)|^2} \leq \prod_{k=1}^n \prod_{i=1}^m \left(\frac{1 + r_{ik}}{1 - r_{ik}} \right)^{w_i}. \quad (14)$$

The first inequality in (14) is untrue in general as it is disproved by substituting Z_1 and Z_2 in (14) with $U|Z_1|$ and $U|Z_2|$, respectively, in the previous example. However, simulation seems to support the second inequality which is unconfirmed yet.

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