Matrix Power Means

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Power Means of Numbers

- The term "power means" is usually used to describe the family of means parametrized by a parameter t and defined by:

$$Q_t(a, b) = \left(\frac{a^t + b^t}{2}\right)^{1/t}$$

  for positive real numbers a and b.
- More generally, if $\mathbb{A}_i = \{a_i : 1 \leq i \leq n\}$ is a set of n positive real numbers and $\{\omega_i : 1 \leq i \leq n\}$ is such that $\sum_i \omega_i = 1$, the weighted power means can be defined by:

$$Q_t(\omega; \mathbb{A}) := \left(\sum_i \omega_i a_i^t\right)^{1/t}.$$

Extension of Power Means to Matrices

- For the rest of the talk, matrices will be assumed to be positive definite unless otherwise specified.
- Power means of numbers can be extended to the set of positive definite matrices in different ways. Naïvely, one would extend the power means as:

$$Q_t(\omega; A, B) = \left(\omega A^t + (1-\omega)B^t\right)^{1/t}.$$

  This extension is well-defined for any $t \in \mathbb{R}$. Some care must be taken when approaching zero, where the limit is considered.
- A less naïve extension is the Kubo-Ando extension of this norm, for $t \in [-1, 1]$

$$P_t(\omega; A, B) = A^{1/2}\left(\omega I + (1-\omega)(A^{-1/2}BA^{-1/2})^t\right)^{1/t} A^{1/2}.$$

Which extension is best?

- Each of the extensions have their own benefits.
- For example, each Kubo-Ando mean has an operator monotone function that generates it. As such, if two means M and N are generated by two functions $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$, respectively, such that $f \geq g$ point-wise, then $M(A, B) \geq N(A, B)$ for all positive definite matrices A and B.
- The power means P_t have been used to study the Karcher means by Lim and Palfia.
- The power means Q_t were used to study the log Euclidean mean by Bhagwat and Subramanian.

Structure of the Talk

- On the first part of the talk we will explore some properties of $P_t(A, B) := P_t(1/2; A, B)$ and their relations with the Arithmetic-Geometric-Harmonic means.
- Then, we will introduce several Arithmetic-Geometric-Harmonic interpolations and compare them to the power means.

Monotonicity of $P_t(\omega; A, B)$

Notice that

$$P_t(\omega; A, B) = A^{1/2} f_t(A^{-1/2} B A^{-1/2}) A^{1/2},$$

where $f_t(x) = (\omega + (1-\omega)x^t)^{1/t}$ for $\omega \in (0,1)$. It is easy to show that f_t is a matrix monotone function on $[0, \infty)$ for $t \in [-1, 1]$. Therefore, if $B \geq D > 0$ we have that

$$A^{-1/2} B A^{-1/2} \geq A^{-1/2} D A^{-1/2}$$

and so $f_t(A^{-1/2} B A^{-1/2}) \geq f_t(A^{-1/2} D A^{-1/2})$. Consequently, for $D \geq B$

$$P_t(\omega; A, B) \geq P_t(\omega; A, D).$$

More is true, if $A \geq C$ and $B \geq D$,

$$P_t(\omega; A, B) \geq P_t(\omega; C, D).$$

A larger interval of t

While the function f_t is not matrix monotone outside of $[-1, 1]$, it can still be shown to be an increasing function of t on $\mathbb{R}$. Hence, the following inequality holds for $-\infty < t \leq s < \infty$

$$P_t(\omega; A, B) \leq P_s(\omega; A, B).$$

In particular, since $P_{-1}(\omega; A, B) = (\omega A^{-1} + (1-\omega)B^{-1})^{-1}$ corresponds with the weighted harmonic means and $P_1(\omega; A, B) = \omega A + (1-\omega)B$ corresponds with the weighted arithmetic means, the power means provide an interpolation between these two means.

Connection to the Geometric Mean

Not so obviously, at $t = 0$ we consider the limit. Lim and Palfia showed that:

$$\lim_{t\to 0} P_t(\omega; A, B) = A^{1/2}(A^{-1/2}BA^{-1/2})^{1-\omega}A^{1/2} =: A\#_{1-\omega}B,$$

which is the Kubo-Ando extension of the geometric mean for positive definite matrices.
Therefore, for $\omega = 1/2$ the power means also provide a Geometric-Arithmetic mean interpolation. For the rest of the talk we will consider the weight to be $\omega = 1/2$ unless otherwise specified.

Heron Means

The most straight-forward of these Geometric-Arithmetic mean interpolations are the Heron means:

$$H_t(A,B) = t\frac{A+B}{2} + (1-t)A\#B,$$

where $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$.
Since these interpolations are also of Kubo-Ando type, to compare them to the power-means it suffices to compare their generating functions.

Theorem

For $t \in [0, 1/2]$ *and positive matrices* A, B*:*

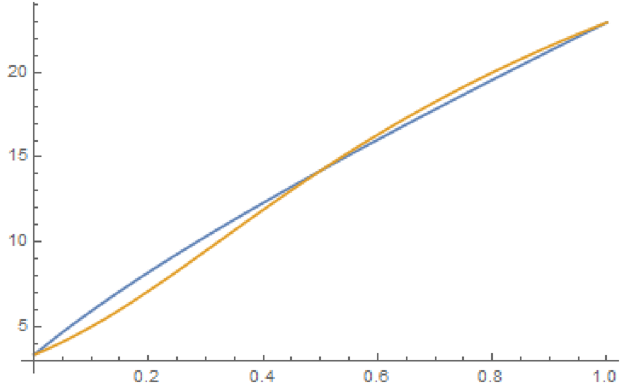
$$H_t(A, B) \geq P_t(A, B).$$

The reversed inequality holds for $t \in [1/2, 0]$*. Equality is obtained only for* $t \in \{0, 1/2, 1\}$*.*

The proof is based on a clever rewriting of the numerical inequality

$$\left(\frac{1 + x^t}{2}\right)^{1/t} \leq t\frac{1 + x}{2} + (1 - t)x^{1/2}$$

Figure: Graph of $\lambda_3(H_t(A,B))$ and $\lambda_3(P_t(A,B))$ on $t \in [0,1]$ for two 3×3 positive definite matrices A and B.



A not so popular interpolation

Similar to the Heron means, we can define the “straight-line" interpolation of the Harmonic-Geometric means

$$F_t(A, B) = t\frac{A!B}{2} + (1-t)A\#B,$$

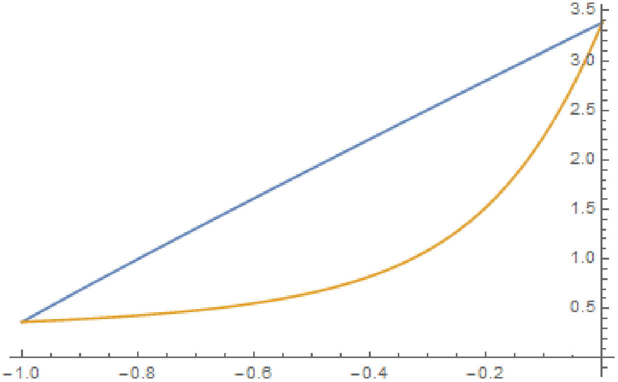
and study the behavior of this interpolation against the behavior of the power means. This result is easy to show, as the power means for numbers are convex:

Theorem

For $t \in [0, 1]$ and positive matrices A, B:

$$F_t(A, B) \geq P_{-t}(A, B).$$

Figure: Graph of $\lambda_3(F_{-t}(A,B))$ and $\lambda_3(P_t(A,B))$ on $t \in [-1, 0]$ for two 3×3 positive definite matrices A and B.



An even less popular interpolation

Now, we define the linear interpolation between the harmonic and arithmetic mean with the parameter $t \in [-1, 1]$:

$$K_t(A, B) := \frac{t+1}{2}\left(\frac{A+B}{2}\right) + \frac{1-t}{2} A!B.$$

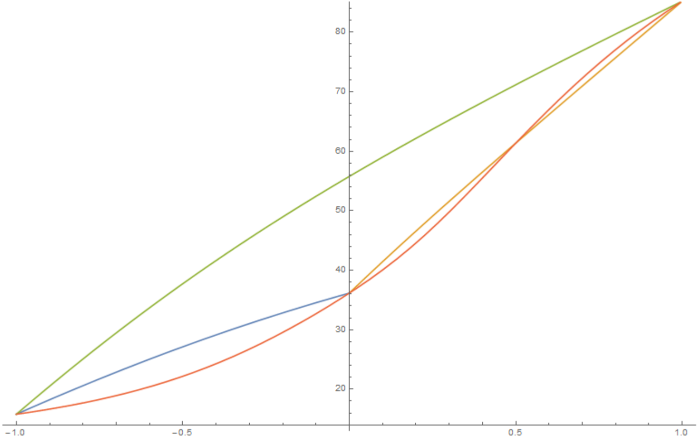
And, we compare this to the other interpolations.

Theorem

Let A and B be positive definite matrices. Then:

1. *For $t \in [-1, 0]$,*
$$K_t(A, B) \geq F_{-t}(A, B).$$
2. *For $t \in [0, 1/2]$,*
$$K_t(A, B) \geq H_t(A, B).$$
3. *For $t \in [1/2, 1]$,*
$$K_t(A, B) \geq P_t(A, B).$$

Figure: Graph of $\lambda_2(K_t(A,B))$ and $\lambda_2(P_t(A,B))$ on $t \in [-1,1]$, $\lambda_2(F_{-t}(A,B))$ on $[-1,0]$, and $\lambda_2(H_t(A,B))$ on $t \in [0,1]$ for two 3×3 positive definite matrices A and B.



A few proofs

For Item 3, the result follows from the inequality for $x > 0$ and $t \in [1/2, 1]$:

$$\frac{t+1}{2}\left(\frac{1+x}{2}\right) + (1-t)\left(\frac{x}{x+1}\right) \geq \left(\frac{1+x^t}{2}\right)^{1/t}.$$

It has been shown that the function on right-hand-side is concave for $t \in [1/2, 1]$. Thus, it is bounded above by its tangent line at $t = 1$,

$$y = \frac{1}{2}\left(x \ln x - (x+1)\ln\frac{x+1}{2}\right)(t-1) + \frac{x+1}{2}$$

on the interval $[1/2, 1]$.

A few proofs

Hence, it suffices to show that the following the inequality holds for $x > 0$ and $t \in [1/2, 1]$,

$$\frac{t+1}{2}\left(\frac{1+x}{2}\right) + (1-t)\left(\frac{x}{x+1}\right) \\ \geq \frac{1}{2}\left(x \ln x - (x+1)\ln\frac{x+1}{2}\right)(t-1) + \frac{x+1}{2}.$$

After several simplifications and arguments using the derivative, this reduces to showing

$$(x-1)^2 \geq 0,$$

which is obviously true.

With a somewhat unusual notation, in this section we will denote the Kubo-Ando extension of the Heinz means for matrices by $G_t(A, B)$

$$G_t(A,B) = 1/2(A\#_t B + A\#_{1-t}B).$$

Theorem

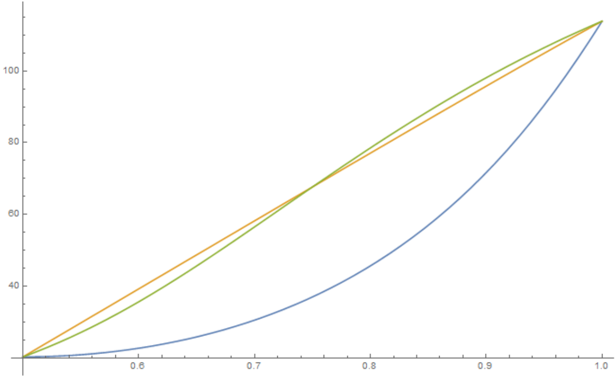
For positive definite matrices A and B and for $t \in [1/2, 3/4]$

$$G_t(A,B) \le P_{2t-1}(A,B) \le H_{2t-1}(A,B)$$

and for $t \in [3/4, 1]$,

$$G_t(A,B) \le H_{2t-1}(A,B) \le P_{2t-1}(A,B)$$

Figure: Graph of $\lambda_2(G_t(A,B))$, $\lambda_2(H_{2t-1}(A,B))$, and $\lambda_2(P_{2t-1}(A,B))$ on $t \in [1/2, 1]$ for two 3×3 positive definite matrices A and B.



One last proof

I will only show,

$$G_t(A, B) \leq H_{2t-1}(A, B),$$

for $t \in [1/2, 1]$. This reduces to show the following inequality,

$$\frac{2t-1}{2}(1+x) + (2-2t)x^{1/2} \geq \frac{x^t + x^{1-t}}{2} \qquad (1)$$

By dividing by $x^{1/2}$ and substituting x by e^y, (1) becomes

$$(2t-1)\cosh\left(\frac{y}{2}\right) + (2-2t) \geq \cosh\left(\frac{y}{2}(2t-1)\right).$$

One last proof

By the concavity of the function $x \mapsto x^{2t-1}$ on this interval,

$$\cosh\left(\frac{y}{2}\right)^{2t-1} = \left(\frac{e^{y/2}+e^{-y/2}}{2}\right)^{2t-1}$$
$$\geq \left(\frac{e^{(2t-1)y/2}+e^{-(2t-1)y/2}}{2}\right) = \cosh\left(\frac{y}{2}(2t-1)\right).$$

So, the desired inequality follows if we show:

$$(2t-1)\cosh\left(\frac{y}{2}\right) + (2-2t) \geq \cosh\left(\frac{y}{2}\right)^{2t-1}.$$

One last proof

Equivalently,

$$za + (1 - z) \geq a^z$$

or

$$z(a - 1) + 1 \geq a^z$$

for any positive real a and $0 \leq z \leq 1$. However, this is just Bernoulli's inequality,

$$(x + 1)^r \leq 1 + rx$$

for $x = a + 1$ and $z = r$. This completes the proof.

Thank you.

Thank you! Questions?