Continuous Dependence and Differentiation of Solutions of a Second Order Boundary Value Problem with an Average Value Condition

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The Boundary Value Problem

Our concern is characterizing derivatives of solutions to the second order nonlocal boundary value problem

$$y'' = f(x, y, y'),\ a < x < b, \tag{1}$$

satisfying

$$y(x_1) = y_1,\ \frac{1}{d-c}\int_c^d y(x)dx = y_2, \tag{2}$$

where $a < x_1 < c < d < b$, and $y_1, y_2 \in \mathbb{R}$ with respect to the boundary parameters.

The Variational Equation

Definition 1.1

Given a solution $y(x)$ of (1), we define the *variational equation along* $y(x)$ by

$$z'' = \frac{\partial f}{\partial y}(x, y(x), y'(x))z + \frac{\partial f}{\partial y'}(x, y(x), y'(x))z'. \tag{3}$$

Assumptions on Equation (1)

We require that

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(ii) for $i = 0, 1$, $\dfrac{\partial f}{\partial y^{(i)}}(x, y, y') : (a, b) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, and

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(iii) solutions of initial value problems for (1) extend to (a, b).

Uniqueness for Equation (1)

(iv) Let $a < x_1 < c < d < b$. If $y(x)$ and $z(x)$ are solutions of (1) where $y(x_1) = z(x_1)$ and $\frac{1}{d-c}\int_c^d y(x)dx = \frac{1}{d-c}\int_c^d z(x)dx$, then, on (a, b),

$$y(x) \equiv z(x).$$

Uniqueness for the Variational Equation

(v) Let $a < x_1 < c < d < b$ and $y(x)$ be a solution of (1). If $u(x)$ is a solution of (3) along $y(x)$ where $u(x_1) = 0$ and $\frac{1}{d-c}\int_c^d u(x)dx = 0$, then, on (a, b),

$$u(x) \equiv 0.$$

A Theorem of Peano

Theorem 3.1

Assume that, with respect to (1), *conditions* (i)-(iii) *are satisfied. Let* $x_0 \in (a, b)$ *and* $y(x) := y(x, x_0, c_1, c_2)$ *denote the solution of* (1) *satisfying the initial conditions* $y(x_0) = c_1$, $y'(x_0) = c_2$.

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(a) *for* $i = 1, 2$, $\alpha_i(x) := \dfrac{\partial y}{\partial c_i}(x)$ *exists on* (a, b) *and is the solution of the variational equation* (3) *along* $y(x)$ *satisfying the respective initial conditions*

$$\alpha_1(x_0) = 1, \; \alpha_1'(x_0) = 0,$$

$$\alpha_2(x_0) = 0, \; \alpha_2'(x_0) = 1.$$

(*b*) $\beta(x) := \dfrac{\partial y}{\partial x_0}(x)$ exists on (a, b) and is the solution of the variational equation (3) along $y(x)$ satisfying the initial conditions

$$\beta(x_0) = -y'(x_0), \;\; \beta'(x_0) = -y''(x_0).$$

(b) $\beta(x) := \dfrac{\partial y}{\partial x_0}(x)$ exists on (a, b) and is the solution of the variational equation (3) along $y(x)$ satisfying the initial conditions

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(c) $\dfrac{\partial y}{\partial x_0}(x) = -y'(x_0)\dfrac{\partial y}{\partial c_1}(x) - y''(x_0)\dfrac{\partial y}{\partial c_2}(x).$

Continuous Dependence for BVPs

Theorem 3.2

Assume (i)-(iv) *are satisfied with respect to* (1)*. Let* $y(x)$ *be a solution of* (1) *on* (a, b)*, and let* $a < \alpha < x_1 < c < d < \beta < b$ *and* $y_1, y_2 \in \mathbb{R}$ *be given. Then, there exists a* $\delta > 0$ *such that, for* $|x_1 - t_1| < \delta$, $|c - \xi| < \delta$, $|d - \Delta| < \delta$, $|y(x_1) - y_1| < \delta$, *and* $|\frac{1}{d-c}\int_c^d y(x)dx - y_2| < \delta$, *there exists a unique solution* $y_\delta(x)$ *of* (1) *such that* $y_\delta(t_1) = y_1$ *and* $\dfrac{1}{\Delta - \xi}\displaystyle\int_\xi^\Delta y_\delta(x)dx = y_2$ *and, for* $i = 0, 1$, $\{y_\delta^{(i)}(x)\}$ *converges uniformly to* $y^{(i)}(x)$ *as* $\delta \to 0$ *on* $[\alpha, \beta]$.

Continuous Dependence for BVPs

Theorem 3.2

Assume (i)-(iv) *are satisfied with respect to* (1). *Let* $y(x)$ *be a solution of* (1) *on* (a, b), *and let* $a < \alpha < x_1 < c < d < \beta < b$ *and* $y_1, y_2 \in \mathbb{R}$ *be given. Then, there exists a* $\delta > 0$ *such that, for* $|x_1 - t_1| < \delta$, $|c - \xi| < \delta$, $|d - \Delta| < \delta$, $|y(x_1) - y_1| < \delta$, *and* $|\frac{1}{d-c}\int_c^d y(x)dx - y_2| < \delta$, *there exists a unique solution* $y_\delta(x)$ *of* (1) *such that* $y_\delta(t_1) = y_1$ *and* $\dfrac{1}{\Delta - \xi}\displaystyle\int_\xi^\Delta y_\delta(x)dx = y_2$ *and, for* $i = 0, 1$, $\{y_\delta^{(i)}(x)\}$ *converges uniformly to* $y^{(i)}(x)$ *as* $\delta \to 0$ *on* $[\alpha, \beta]$.

For a typical proof, we refer [4].

Main Result

Theorem 4.1

Assume conditions (i)-(v) *are satisfied. Let* $y(x)$ *be a solution of* (1) *on* (a, b). *Let* $a < x_1 < c < d < b$ *and* $y_1, y_2 \in \mathbb{R}$ *be given so that*

$$y(x) = y(x, x_1, y_1, y_2, c, d),$$

where

$$y(x_1) = y_1, \ \frac{1}{d-c}\int_c^d y(x)dx = y_2.$$

Then,

(a) for $i = 1, 2$, $u_i(x) := \dfrac{\partial y}{\partial y_i}(x)$ exists on (a, b) and is the solution of the variational equation (3) along $y(x)$ satisfying the respective boundary conditions

$$u_1(x_1) = 1 \text{ and } \frac{1}{d-c}\int_c^d u_1(x)dx = 0,$$

$$u_2(x_1) = 0 \text{ and } \frac{1}{d-c}\int_c^d u_2(x)dx = 1,$$

Then,

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$$u_2(x_1) = 0 \text{ and } \frac{1}{d-c}\int_c^d u_2(x)dx = 1,$$

(b) $z(x) := \dfrac{\partial y}{\partial x_1}(x)$ exists on (a, b) and is the solution of the variational equation (3) along $y(x)$ satisfying the respective boundary conditions

$$z(x_1) = -y'(x_1) \text{ and } \frac{1}{d-c}\int_c^d z(x)dx = 0,$$

(c) $C(x) := \dfrac{\partial y}{\partial c}(x)$ exists on (a, b) and is the solution of the variational equation (3) along $y(x)$ satisfying the boundary conditions

$$C(x_1) = 0 \text{ and } \frac{1}{d-c}\int_c^d C(x)dx = \frac{y(c) - y_2}{d-c}, \text{and}$$

(c) $C(x) := \dfrac{\partial y}{\partial c}(x)$ exists on (a, b) and is the solution of the variational equation (3) along $y(x)$ satisfying the boundary conditions

$$C(x_1) = 0 \text{ and } \frac{1}{d-c}\int_c^d C(x)dx = \frac{y(c) - y_2}{d-c}, \text{and}$$

(d) $D(x) := \dfrac{\partial y}{\partial d}(x)$ exists on and is the solution of the variational equation (3) along $y(x)$ satisfying the boundary conditions

$$D(x_1) = 0 \text{ and } \frac{1}{d-c}\int_c^d D(x)dx = \frac{y_2 - y(d)}{d-c}.$$

Proof of Part (d)

Denote $y(x, x_1, y_1, y_2, c, d)$ by $y(x, d)$.

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Let $\delta > 0$ be as in the Continuous Dependence Theorem, $0 < |h| < \delta$ be given, and define the difference quotient

$$D_h(x) = \frac{1}{h}[y(x, d+h) - y(x, d)].$$

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Show that $D_h(x)$:

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Show that $D_h(x)$:

- Satisfies the boundary conditions.
- Solves the variational equation (3).
- Has a limit as $h \to 0$.

Note that for every $h \neq 0$,

$$D_h(x_1) = \frac{1}{h}[y(x_1, d+h) - y(x_1, d)] = \frac{1}{h}[y_1 - y_1] = 0$$

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and

$$\frac{1}{d-c}\int_c^d D_h(x)dx = \frac{1}{d-c}\int_c^d \frac{y(x, d+h) - y(x, d)}{h}dx$$

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$$D_h(x_1) = \frac{1}{h}[y(x_1, d+h) - y(x_1, d)] = \frac{1}{h}[y_1 - y_1] = 0$$

and

$$\begin{aligned} \frac{1}{d-c}\int_c^d D_h(x)dx &= \frac{1}{d-c}\int_c^d \frac{y(x,d+h)-y(x,d)}{h}dx \\ &= \frac{1}{d-c}\int_c^d \frac{y(x,d+h)}{h}dx - \frac{1}{d-c}\int_c^d \frac{y(x,d)}{h}dx \end{aligned}$$

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and

$$\begin{aligned}
\frac{1}{d-c}\int_c^d D_h(x)dx &= \frac{1}{d-c}\int_c^d \frac{y(x,d+h) - y(x,d)}{h}dx \\
&= \frac{1}{d-c}\int_c^d \frac{y(x,d+h)}{h}dx - \frac{1}{d-c}\int_c^d \frac{y(x,d)}{h}dx \\
&= \frac{1}{d-c}\left[\int_c^{d+h} \frac{y(x,d+h)}{h}dx + \int_{d+h}^d \frac{y(x,d+h)}{h}dx\right] - \frac{y_2}{h}
\end{aligned}$$

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\frac{1}{d-c}\int_c^d D_h(x)dx &= \frac{1}{d-c}\int_c^d \frac{y(x,d+h)-y(x,d)}{h}dx \\
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&= \frac{1}{d-c}\left[\int_c^{d+h} \frac{y(x,d+h)}{h}dx + \int_{d+h}^d \frac{y(x,d+h)}{h}dx\right] - \frac{y_2}{h} \\
&= \frac{1}{d-c}\frac{(d+h)-c}{(d+h)-c}\int_c^{d+h} \frac{y(x,d+h)}{h}dx \\
&+ \frac{y(e)(d-(d+h))}{h(d-c)} - \frac{y_2}{h}
\end{aligned}$$

Note that for every $h \neq 0$,

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&= \frac{1}{d-c}\frac{(d+h)-c}{(d+h)-c}\int_c^{d+h} \frac{y(x,d+h)}{h}dx \\
&+ \frac{y(e)(d-(d+h))}{h(d-c)} - \frac{y_2}{h} \\
&= \frac{((d+h)-c)y_2}{h(d-c)} - \frac{y(e)}{d-c} - \frac{y_2(d-c)}{h(d-c)}
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&= \frac{1}{d-c}\frac{(d+h)-c}{(d+h)-c}\int_c^{d+h} \frac{y(x,d+h)}{h}dx \\
&+ \frac{y(e)(d-(d+h))}{h(d-c)} - \frac{y_2}{h} \\
&= \frac{((d+h)-c)y_2}{h(d-c)} - \frac{y(e)}{d-c} - \frac{y_2(d-c)}{h(d-c)} \\
&= \frac{y_2 - y(e)}{d-c} \text{ for some } e \text{ between } d \text{ and } d+h.
\end{aligned}$$

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To that end, let

$$\mu = y'(x_1, d)$$

and

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Then,

$$y(x) = u(x, x_1, y_1, \mu),$$

and we have

$$D_h(x) = \frac{1}{h}[u(x, x_1, y_1, \mu + \nu) - u(x, x_1, y_1, \mu)].$$

By the Mean Value Theorem, we obtain

$$D_h(x) = \frac{1}{h}\left[\frac{\partial u}{\partial \mu}(x, u(x, x_1, y_1, \mu + \bar{\nu}))(\mu + \nu - \mu)\right]$$

where $\mu + \bar{\nu}$ is between μ and $\mu + \nu$.

By the Mean Value Theorem, we obtain

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where $\mu + \bar{\nu}$ is between μ and $\mu + \nu$.

Then, using the notation from Peano's Theorem,

$$D_h(x) = \frac{\nu}{h}\alpha_2(x, u(x, x_1, y_1, \mu + \bar{\nu})),$$

where $\alpha_2(x, u(\cdot))$ is the solution of the variational equation (3) along $u(\cdot)$ satisfying

$$\alpha_2(x_1) = 0, \;\; \alpha_2'(x_1) = 1.$$

By hypothesis (v), the fact that $\alpha_2(x, u(\cdot))$ is a nontrivial solution of (3) along $u(\cdot)$, and $\alpha_2(x_1, u(\cdot)) = 0$, we have $\frac{1}{d-c}\int_c^d \alpha_2(x, u(\cdot))dx \neq 0$.

By hypothesis (v), the fact that $\alpha_2(x, u(\cdot))$ is a nontrivial solution of (3) along $u(\cdot)$, and $\alpha_2(x_1, u(\cdot)) = 0$, we have $\frac{1}{d-c}\int_c^d \alpha_2(x, u(\cdot))dx \neq 0$.

Recall,

$$\frac{1}{d-c}\int_c^d D_h(x)dx = \frac{y_2 - y(e)}{d-c}.$$

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Recall,

$$\frac{1}{d-c}\int_c^d D_h(x)dx = \frac{y_2 - y(e)}{d-c}.$$

Therefore,

$$\frac{1}{d-c}\int_c^d \frac{\nu}{h}\alpha_2(x, u(x, x_1, y_1, \mu + \bar{\nu})) = \frac{y_2 - y(e)}{d-c}.$$

By hypothesis (v), the fact that $\alpha_2(x, u(\cdot))$ is a nontrivial solution of (3) along $u(\cdot)$, and $\alpha_2(x_1, u(\cdot)) = 0$, we have $\frac{1}{d-c}\int_c^d \alpha_2(x, u(\cdot))dx \neq 0$.

Recall,

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Therefore,

$$\frac{1}{d-c}\int_c^d \frac{\nu}{h}\alpha_2(x, u(x, x_1, y_1, \mu + \bar{\nu})) = \frac{y_2 - y(e)}{d-c}.$$

Solving for $\frac{\nu}{h}$ and taking the limit yields

$$\begin{aligned}\lim_{h\to 0}\frac{\nu}{h} &= \lim_{h\to 0}\frac{\frac{y_2 - y(e)}{d-c}}{\frac{1}{d-c}\int_c^d \alpha_2(x, u(\cdot))dx} \\ &= \frac{y_2 - y(d)}{\int_c^d \alpha_2(x, u(\cdot))dx} := M.\end{aligned}$$

Now let

$$D(x) = \lim_{h \to 0} D_h(x),$$

and note by construction of $D_h(x)$,

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Furthermore,

$$D(x) = \lim_{h \to 0} D_h(x) = \lim_{h \to 0} \frac{\nu}{h} \alpha_2(x, u(\cdot)) = M\alpha_2(x, y(x))$$

which is a solution of the variational equation (3) along $y(x)$.

In addition,

$$D(x_1) = \lim_{h\to 0} D_h(x_1) = 0,$$

and

$$\begin{aligned}
\frac{1}{d-c}\int_c^d D(x)dx &= \lim_{h\to 0}\left[\frac{1}{d-c}\int_c^d D_h(x)dx\right] \\
&= \lim_{h\to 0}\frac{y_2 - y(e)}{d-c} \\
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$$\begin{aligned} \frac{1}{d-c}\int_c^d D(x)dx &= \lim_{h \to 0}\left[\frac{1}{d-c}\int_c^d D_h(x)dx\right] \\ &= \lim_{h \to 0}\frac{y_2 - y(e)}{d-c} \\ &= \frac{y_2 - y(d)}{d-c}. \end{aligned}$$

This completes the argument for part (d).

Analogue to Part (c) of Peano

Corollary 6.1

Under the assumptions of the Main Result Theorem, we have

(a) $$\frac{\partial y}{\partial x_1} = -y'(x_1)\frac{\partial y}{\partial y_1},$$

Analogue to Part (c) of Peano

Corollary 6.1

Under the assumptions of the Main Result Theorem, we have

(a) $\dfrac{\partial y}{\partial x_1} = -y'(x_1)\dfrac{\partial y}{\partial y_1}$,

(b) $\dfrac{\partial y}{\partial c} = -\dfrac{y_2 - y(c)}{y_2 - y(d)} \cdot \dfrac{\partial y}{\partial d}$.

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THANK YOU!