Periodicity in Quantum Calculus

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For $q > 1$, the time scale

$$q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\} = \{1, q, q^2, q^3, q^4, \dots\},$$

is called the quantum time scale.

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- On $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$, $\rho(t) = t$ and $\mu(t) \equiv 0$.
- On $T = q^{\mathbb{N}_0}$, $\sigma(t) = qt$, $\rho(t) = \dfrac{t}{q}$ $(t > 1)$, and $\mu(t) = (q - 1)t$.

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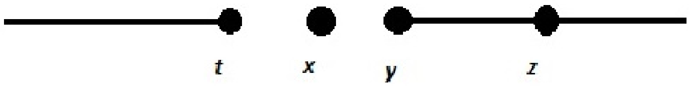
- If $\rho(t) = t = \sigma(t)$, t is dense.

Consider the time scale $\mathbb{T}$ given by



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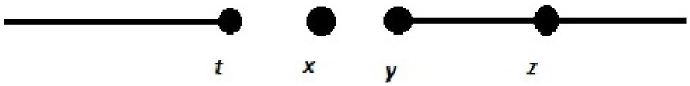

- t is left-dense and right-scattered.
- x is isolated.
- y is left-scattered and right-dense.
- z is dense.

Definition 1.4

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T} \setminus \sup \mathbb{T}$. Then if f is continuous at t and t is right-scattered ($t < \sigma(t)$), then f is differentiable at t with

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$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Notice this is the slope of the secant line connecting $(t, f(t))$ and $(\sigma(t), f(\sigma(t)))$.

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- On $\mathbb{T} = q^{N_0}$,

$$f^{\Delta}(t) = D_q f(t) = \frac{f(qt) - f(t)}{(q-1)t}.$$

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Notice the similarity between f^{Δ} and f'.

- On $\mathbb{T} = \mathbb{R}$,

$$f^{\Delta}(t) = f'(t)$$

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Here $f^{\sigma}(t) = f(\sigma(t))$.

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Similarly, on a general time scale $\mathbb{T}$, we define $e_a(t, t_0)$ as the unique solution of the initial value problem

$$y^{\Delta} = ay, \ y(t_0) = 1.$$

Let's find $e_1(t, 0)$ on $\mathbb{T} = \mathbb{Z}$.

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iff $a-1=1$, or $a=2$. So $e_1(t,0)=2^t$ on $\mathbb{T}=\mathbb{N}$.

- On $2^{\mathbb{N}_0}$,

$$e_1(t,1) = \prod_{s\in[1,t)\cap\mathbb{T}} (1+s) = 2\cdot 3\cdot 5\cdot\cdots\cdot\left(\frac{t}{2}+1\right).$$

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So, for example,

$$e_1(2,1) = 2,\ \ e_1(4,1) = 2\cdot 3 = 6,\ \ e_1(8,1) = 2\cdot 3\cdot 5 = 30.$$

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$$e_1^{\Delta}(2,1) = \frac{6-2}{2} = 2 = e_1(2,1),$$

and

$$e_1^{\Delta}(4,1) = \frac{30-6}{4} = 6 = e_1(4,1).$$

Definition 1.7

If $F^{\Delta}(t) = f(t)$, define the indefinite integral of f by

$$\int f(t)\Delta t = F(t) + C,$$

and the definite integral of f from a to b by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

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$$\int_{q^m}^{q^n} f(s)\Delta s = \int_{q^m}^{q^n} f(s) d_q s = (q-1)\sum_{k=m}^{n-1} q^k f(q^k). \qquad (1)$$

Definition 2.1

A time scale, denoted by $\mathbb{T}$, is said to be additively T-periodic if there exists a $T > 0$ such that $t \pm T \in \mathbb{T}$ for all $t \in \mathbb{T}$ (see [3]).

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A time scale, denoted by $\mathbb{T}$, is said to be additively T-periodic if there exists a $T > 0$ such that $t \pm T \in \mathbb{T}$ for all $t \in \mathbb{T}$ (see [3]). A function f on additively periodic domain $\mathbb{T}$ with period T is said to be periodic with period P if there exists an $n \in \mathbb{N}$ such that $P = nT$, $f(t \pm P) = f(t)$ for all $t \in \mathbb{T}$, and P is the smallest number such that $f(t \pm P) = f(t)$.

Notice, since the time scale $q^{\mathbb{N}_0}$ is not additive, we cannot define periodicity on $q^{\mathbb{N}_0}$ in a same way we do on additively periodic domains. Recently, two different definitions of periodicity on $q^{\mathbb{N}_0}$ have arose. We wish to show a relationship between these two definitions of periodicity. We will then show the existence of P-periodic solutions of a q-Volterra integral equation.

Definition 2.2 (Bohner and Chieochan, [2])

Let $P \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is said to be P-periodic if

$$f(t) = q^P f\left(q^P t\right) \text{ for all } t \in q^{\mathbb{N}_0}. \tag{2}$$

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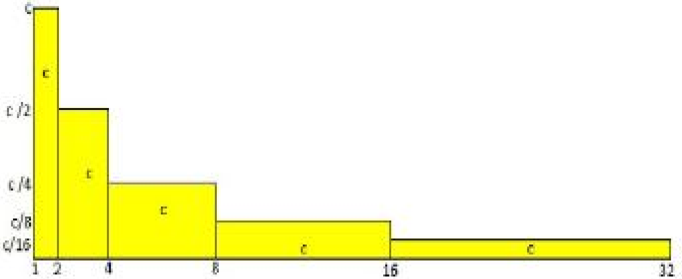
Periodicity in this definition is based on the equality of areas lying below the graph of the function at each period.



Definition 2.3 (Adivar, [1])

Let $P \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is said to be P-periodic if

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This definition regards a periodic function to be the one repeating its values after a certain number of steps on $q^{\mathbb{N}_0}$.

For example, the function

$$h(t) = (-1)^{\frac{\ln t}{\ln q}}$$

on $q^{\mathbb{N}_0}$ is a 2-periodic function according to Definition 2.3, since $h(q^2 t) = h(t)$ holds.

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$$g(t) = 1/t$$

is 1-periodic with respect to Definition 2.2 since it satisfies $qg(qt) = g(t)$.

Theorem 2.4

Let $f : q^{\mathbb{N}_0} \to \mathbb{R}$. Then f is periodic with respect to of Definition 2.2 if and only if $\tilde{f}(t) = tf(t)$ is periodic with respect to Definition 2.3 with the same period.

Theorem 2.5

The function x is a P-periodic solution of the following first order q-difference equation

$$D_q x(t) + a(t) x^{\sigma}(t) = f\left(t, tx\left(t\right)\right), \ t \in q^{\mathbb{N}_0}, \tag{3}$$

with respect to Definition 2.2 if and only if $\tilde{x}(t) := tx(t)$ is a P-periodic solution of the first order q-difference equation

$$D_q \tilde{x}(t) + \tilde{a}(t) \tilde{x}^{\sigma}(t) = \tilde{f}\left(t, \tilde{x}\left(t\right)\right), \ t \in q^{\mathbb{N}_0}, \tag{4}$$

where

$$\tilde{a}(t) := \frac{ta\left(t\right) - 1}{qt},$$

and $\tilde{f}\left(t, \tilde{x}\left(t\right)\right) = tf\left(t, tx\left(t\right)\right)$, with respect to Definition 2.3.

Proof:

Let x be a solution of (3). Then

$$tD_q x(t) + x^{\sigma}(t) - x^{\sigma}(t) + ta(t)x^{\sigma}(t) = tf\left(t, tx\left(t\right)\right),$$

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$$D_q(tx(t)) + \frac{ta(t) - 1}{qt} qtx^\sigma(t) = tf(t, tx(t)).$$

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This implies $\tilde{x}$ solves (4). The proof that $\tilde{x}$ solves (4) implies x solves (3) is similar. Theorem 2.4 implies that x is P-periodic with respect to Definition 2.2 if and only if $\tilde{x}$ is P-periodic with respect to Definition 2.3.

Suppose that $a : q^{\mathbb{N}_0} \to \mathbb{R}$ is a function with $(1 + (q-1)ta(t)) \neq 0$ for all $t \in q^{\mathbb{N}_0}$. Based on the function a, we define the natural exponential functions

$$e_a(q^n, q^m) := \prod_{k=m}^{n-1} (1 + (q-1)q^k a(q^k)) \text{ and } e_{\ominus a}(q^n, q^m) := e_a(q^n, q^m)^{-1}.$$

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Multiplying both sides of the equations (3) and (4) with $e_a(t,1)$ and $e_{\tilde{a}}(t,1)$, respectively, we obtain the following integral equations

$$x(t) = q^P e_{\ominus a}\left(q^P t, t\right) x(t) + q^P \int_t^{q^P t} e_{\ominus a}\left(q^P t, s\right) f(s, sx(s))\, d_q s, \qquad (5)$$

and

$$\tilde{x}(t) = e_{\ominus \tilde{a}}\left(q^P t, t\right) \tilde{x}(t) + \int_t^{q^P t} e_{\ominus \tilde{a}}\left(q^P t, s\right) f(s, \tilde{x}(s))\, d_q s, \qquad (6)$$

for $t \in q^{\mathbb{N}_0}$.

Next, the generalizations of (5) and (6) have the form of q-Volterra integral equations as follows:

$$x(t) = g\left(t, tx\left(t\right)\right) + \int_{t}^{q^P t} C\left(t,s\right) f\left(s, sx\left(s\right)\right) d_q s, \tag{7}$$

and

$$\tilde{x}(t) = \tilde{g}\left(t, \tilde{x}\left(t\right)\right) + \int_{t}^{q^P t} \tilde{C}\left(t,s\right) \tilde{f}\left(s, \tilde{x}\left(s\right)\right) d_q s, \tag{8}$$

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where $g, \tilde{g}, f, \tilde{f} : q^{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ are continuous in their second variable and $C, \tilde{C} : q^{\mathbb{N}_0} \times q^{\mathbb{N}_0} \to \mathbb{R}$,

$$\tilde{C}\left(t, s\right) = \frac{t}{s} C\left(t, s\right), \tag{9}$$

$$\tilde{f}\left(t, \tilde{x}\left(t\right)\right) = tf\left(t, tx\left(t\right)\right), \tag{10}$$

and

$$\tilde{g}\left(t, \tilde{x}\left(t\right)\right) = tg\left(t, tx\left(t\right)\right). \tag{11}$$

Theorem 2.6

Assume that C, f, g and x satisfy

$$C\left(q^P t, q^P s\right) = C(t, s), \tag{12}$$

$$q^P f\left(q^P t, q^P tx\left(q^P t\right)\right) = f\left(t, tx\left(t\right)\right), \tag{13}$$

$$q^P g\left(q^P t, q^P tx\left(q^P t\right)\right) = g\left(t, tx\left(t\right)\right). \tag{14}$$

Theorem 2.6

Assume that C, f, g *and* x *satisfy*

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$$q^P g\left(q^P t, q^P tx\left(q^P t\right)\right) = g\left(t, tx\left(t\right)\right). \tag{14}$$

Then $x(t)$ *is a periodic solution of* (7) *with respect to Definition 2.2 if and only if* $\tilde{x}\left(t\right) = tx\left(t\right)$ *is a periodic solution of* (8) *with respect to Definition 2.3.*

Proof: Assume (12)-(14) hold and suppose that $x(t)$ solves (7) and is P-periodic with respect to Definition 2.2. Let us multiply both sides of (7) by t, i.e.,

$$tx(t) = tg\left(t, tx\left(t\right)\right) + \int_{t}^{q^P t} tC\left(t, s\right) f\left(s, sx\left(s\right)\right) d_q s,$$

or

$$tx(t) = tg\left(t, tx\left(t\right)\right) + \int_{t}^{q^P t} \frac{t}{s} C\left(t, s\right) sf\left(s, sx\left(s\right)\right) d_q s.$$

By employing (9)-(11), we get

$$\tilde{x}(t) = \tilde{g}\left(t, \tilde{x}\left(t\right)\right) + \int_{t}^{q^P t} \tilde{C}\left(t, s\right) \tilde{f}\left(s, \tilde{x}\left(s\right)\right) d_q s. \tag{15}$$

Notice that $\tilde{x}(t)$ is a P-periodic solution of (15) with respect to Definition 2.3. To show this, consider

$$\tilde{x}(q^P t) = q^P tx\left(q^P t\right) = \tilde{g}\left(q^P t, \tilde{x}\left(q^P t\right)\right) + \int_{q^P t}^{q^{2P} t} \tilde{C}\left(q^P t, s\right) \tilde{f}\left(s, \tilde{x}\left(s\right)\right) d_q s$$

$$= \tilde{g}\left(q^P t, \tilde{x}\left(q^P t\right)\right) + \int_{t}^{q^P t} \tilde{C}\left(q^P t, q^P s\right) \tilde{f}\left(q^P s, \tilde{x}\left(q^P s\right)\right) d_q s$$

$$= q^P tg\left(q^P t, q^P tx\left(q^P t\right)\right)$$

$$+ \int_{t}^{q^P t} \frac{q^P t}{q^P s} C\left(q^P t, q^P s\right) q^P s \; f\left(q^P s, q^P sx\left(q^P s\right)\right) d_q s.$$

Using (12)-(14) we get

$$\tilde{x}(q^P t) = tg\left(t, tx\left(t\right)\right) + \int_t^{q^P t} \frac{t}{s} C\left(t, s\right) s \; f\left(s, sx\left(s\right)\right) d_q s$$

$$= \tilde{g}\left(t, \tilde{x}\left(t\right)\right) + \int_t^{q^P t} \tilde{C}\left(t, s\right) \tilde{f}\left(s, \tilde{x}\left(s\right)\right) d_q s.$$

The proof of the necessity part can be done by following a similar procedure used in the sufficiency part, hence, we omit it.

We study the existence of periodic solutions of the following type q-Volterra integral equations

$$\tilde{x}(t) = \tilde{g}\left(t, \tilde{x}\left(t\right)\right) + \int_{t}^{q^{P}t} \tilde{C}\left(t, s\right) \tilde{f}\left(s, \tilde{x}\left(s\right)\right) d_q s, \qquad (16)$$

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Let $\mathbb{P}$ be the set of all functions defined on $q^{\mathbb{N}_0}$ which are P-periodic. Then $(\mathbb{P}, \|\cdot\|)$ is a Banach space endowed with the norm

$$\|x\| = \max_{t \in \left[1, q^P\right]_{q^{\mathbb{N}_0}}} |x(t)|$$

where $\left[1, q^P\right]_{q^{\mathbb{N}_0}} := \left[1, q^P\right] \cap q^{\mathbb{N}_0}$.

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$\mathcal{K}1$ $\tilde{g}$ satisfies

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$$\tilde{C}\left(q^P t, q^P s\right) = \tilde{C}\left(t, s\right),$$

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$$q^P \tilde{f}\left(q^P t, \tilde{x}\right) = \tilde{f}\left(t, \tilde{x}\right),$$

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for all $t \in q^{\mathbb{N}_0}$; and

$\mathcal{K}4$ $\left|\tilde{g}\left(t, \tilde{x}\right) - \tilde{g}\left(t, \tilde{y}\right)\right| \leq a_1 \left|\tilde{x} - \tilde{y}\right|,\ a_1 \in (0, 1)$.

Lemma 3.1

Assume ($\mathcal{K}$1-$\mathcal{K}$3) and for $\tilde{\varphi} \in \mathbb{P}$ define the operator Q as

$$(Q\tilde{\varphi})(t) := \tilde{g}(t, \tilde{\varphi}(t)) + \int_{t}^{q^P t} \tilde{C}(t,s)\, \tilde{f}(s, \tilde{\varphi}(s))\, d_q s. \tag{17}$$

Then $Q : \mathbb{P} \to \mathbb{P}$.

Theorem 3.2 (Krasnosel'skii, [4])

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Theorem 3.2 (Krasnosel'skii, [4])

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(i) $x, y \in \mathbb{M}$ *implies* $Ax + By \in \mathbb{M}$,

(ii) A *is a contraction mapping, and*

(iii) B *is a compact and continuous mapping.*

Then there exists $z \in \mathbb{M}$ *with* $z = Az + Bz$.

Now, the operator Q given in (17) can be written as

$$(Q\tilde{\varphi})(t) := (A\tilde{\varphi})(t) + (B\tilde{\varphi})(t),$$

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$$(B\tilde{\varphi})(t) := \int_t^{q^P t} \tilde{C}(t,s)\,\tilde{f}(s, \tilde{\varphi}(s))\, d_q s. \tag{19}$$

Lemma 3.3

Suppose ($\mathcal{K}4$) holds. Then $A : \mathbb{P} \to \mathbb{P}$ is a contraction mapping.

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Suppose $(\mathcal{K}4)$ *holds. Then* $A : \mathbb{P} \to \mathbb{P}$ *is a contraction mapping.*

Lemma 3.4

Assume $(\mathcal{K}1\text{-}\mathcal{K}3)$ *hold. Then* $B : \mathbb{P} \to \mathbb{P}$ *is a continuous compact mapping.*

Define

$$\bar{C} := \max_{(t,s)\in\left[1,q^P\right]_{q^{\mathbb{N}_0}}\times\left[t,q^P t\right]_{q^{\mathbb{N}_0}}} \left|\tilde{C}(t,s)\right|. \tag{20}$$

Define

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Define the function $\bar{F} : \mathbb{R} \to \mathbb{R}$ by

$$\bar{F}(m) = \sup_{(t,x)\in[1,q^P]_{q^{\mathbb{N}_0}}\times[-m,m]} |\tilde{f}(t,x)|. \qquad (21)$$

Theorem 3.5

Assume ($\mathcal{K}$1-$\mathcal{K}$4). If there exists a positive constant M_0 such that

$$\frac{\alpha + \bar{C}\bar{F}(M_0)q^P(q^P - 1)}{1 - a_1} \leq M_0, \tag{22}$$

where $\alpha = \|g(t,0)\|$, then equation (16) has a P-periodic solution in $\Pi_{M_0} := \{\varphi \in \mathbb{P} : \|\varphi\| \leq M_0\}$ with respect to Definition 2.3.

Proof:

For $\tilde{\varphi}, \tilde{\psi} \in \Pi_{M_0}$, we have, for $t \in [1, q^P]_{q^{\mathbb{N}_0}}$,

$$|A\tilde{\varphi} + B\tilde{\psi}|(t) \leq |\tilde{g}(t, \tilde{\varphi}(t))| + \left| \int_t^{q^P t} \tilde{C}(t,s)\tilde{f}(s, \tilde{\phi}(s)) d_q s \right|. \quad (23)$$

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Notice that for $t \in [1, q^P]_{q^{\mathbb{N}_0}}$,

$$\begin{aligned} |\tilde{g}(t, \tilde{\varphi}(t))| &\leq |\tilde{g}(t, \tilde{\varphi}(t)) - \tilde{g}(t, 0)| + |g(t, 0)| \\ &\leq a_1 |\tilde{\varphi}(t)| + \alpha \\ &\leq a_1 \|\tilde{\varphi}\| + \alpha \\ &\leq a_1 M_0 + \alpha. \end{aligned} \quad (24)$$

Therefore by (22), (23), and (24), we obtain, for $t \in [1, q^P]_{q^{\mathbb{N}_0}}$,

$$|A\tilde{\varphi} + B\tilde{\psi}|(t) \leq a_1 M_0 + \alpha + \bar{C}\bar{F}(M_0) q^P (q^P - 1) \leq M_0.$$

Therefore by (22), (23), and (24), we obtain, for $t \in [1, q^P]_{q^{\mathbb{N}_0}}$,

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Therefore, for $\tilde{\varphi}, \tilde{\psi} \in \Pi_{M_0}$, $\|A\tilde{\varphi} + B\tilde{\psi}\| \leq M_0$. So $A\tilde{\varphi} + B\tilde{\psi} \in \Pi_{M_0}$, which proves condition (i) of Theorem 3.2.

Therefore by (22), (23), and (24), we obtain, for $t \in [1, q^P]_{q^{\mathbb{N}_0}}$,

$$|A\tilde{\varphi} + B\tilde{\psi}|(t) \leq a_1 M_0 + \alpha + \bar{C}\bar{F}(M_0)q^P(q^P - 1) \leq M_0.$$

Therefore, for $\tilde{\varphi}, \tilde{\psi} \in \Pi_{M_0}$, $\|A\tilde{\varphi} + B\tilde{\psi}\| \leq M_0$. So $A\tilde{\varphi} + B\tilde{\psi} \in \Pi_{M_0}$, which proves condition (i) of Theorem 3.2. Notice Lemma 3.3 and Lemma 3.4 prove conditions (ii) and (iii) of Theorem 3.2.

Therefore by (22), (23), and (24), we obtain, for $t \in [1, q^P]_{q^{\mathbb{N}_0}}$,

$$|A\tilde{\varphi} + B\tilde{\psi}|(t) \leq a_1 M_0 + \alpha + \bar{C}\bar{F}(M_0)q^P(q^P - 1) \leq M_0.$$

Therefore, for $\tilde{\varphi}, \tilde{\psi} \in \Pi_{M_0}$, $\|A\tilde{\varphi} + B\tilde{\psi}\| \leq M_0$. So $A\tilde{\varphi} + B\tilde{\psi} \in \Pi_{M_0}$, which proves condition (i) of Theorem 3.2. Notice Lemma 3.3 and Lemma 3.4 prove conditions (ii) and (iii) of Theorem 3.2. Therefore there exists a P-periodic solution of (16) with respect to Definition 2.3.

Example 4.1

Consider the following equation

$$\tilde{x}(t) = \frac{1}{2}\sin\left(\frac{\ln t}{\ln 2}\pi\right)\tilde{x}(t) + \frac{1}{48e^4}\int_t^{4t}\exp\left(\frac{s}{t}\right)\tilde{\Lambda}(s)\,\tilde{x}(s)d_2 s, \ \ t \in 2^{\mathbb{N}_0}. \tag{25}$$

Here

$$\tilde{\Lambda}(t) = \begin{cases} 1/t, \ \text{if } \log_2 t \text{ is odd} \\ 2/t, \ \text{if } \log_2 t \text{ is even.} \end{cases}$$

$$\tilde{x}(t) = \frac{1}{2}\sin\left(\frac{\ln t}{\ln 2}\pi\right)\tilde{x}(t) + \frac{1}{48e^4}\int_t^{4t} \exp\left(\frac{s}{t}\right)\tilde{\Lambda}(s)\,\tilde{x}(s)d_2 s$$

Here

$$\tilde{g}\left(t, \tilde{x}\left(t\right)\right) = \frac{1}{2}\sin\left(\frac{\ln t}{\ln 2}\pi\right)\tilde{x}(t),$$

$$\tilde{C}\left(t, s\right) = \exp\left(\frac{s}{t}\right),$$

and

$$\tilde{f}\left(t, \tilde{x}\left(t\right)\right) = \frac{1}{48e^4}\tilde{\Lambda}\left(t\right)\tilde{x}(t),$$

Observe that assumptions $(\mathcal{K}1 - \mathcal{K}3)$ are satisfied, and the function $\tilde{g}$ satisfies the Lipschitz condition $(\mathcal{K}4)$ with constant $a_1 = 1/2$. We obtain $\bar{C} = e^4$ and $\bar{F}\left(M_0\right) = (1/e^4 24)M_0$, respectively. Also, $\alpha = 0$. Then, the inequality (22) is satisfied for any positive constant M_0. By Theorem 3.5, we conclude that the equation (25) has a 2-periodic solution with respect to Definition 2.3.

Example 4.2

Since (25) has a 2-periodic solution with respect to Definition 2.3, then the integral equation

$$x(t) = \frac{1}{2}\sin\left(\frac{\ln t}{\ln 2}\pi\right)x(t) + \frac{1}{48e^4}\int_t^{4t}\frac{1}{t}\exp\left(\frac{s}{t}\right)\tilde{\Lambda}(s)sx(s)d_2s, \tag{26}$$

has a 2-periodic solution with respect to Definition 2.2.

THANK YOU!

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