

STABILITY OF NON-UNIQUE SOLUTIONS OF DIFFERENTIAL EQUATIONS

MUHAMMAD N. ISLAM

ABSTRACT. The existence of asymptotically stable solutions of the following system of nonlinear differential equations

$$x'(t) + A(t)x(t) = f(t, x(t)), \quad x(0) = x_0, \quad t \geq 0,$$

is studied in this paper. Schauder's fixed point theorem is used in the analysis. The stability that is studied in this paper is not the standard Liapunov stability, which is commonly studied by the researchers in the field of differential equations.

1. Introduction

J. Banas and R. Rzepka, An application of a measure of noncompactness in the study of asymptotic stability, Appl. Math. Lett. 16, (2003), 1-6.

T.A.Burton and Bo Zhang, Fixed points and stability of an integral equation: Nonuniqueness, Appl. Math. Lett. 17 (2004), 839-846.

Let n be a positive integer. Consider following initial value problem (I.V.P.):

$$x'(t) + A(t)x(t) = f(t, x(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (1)$$

where $x = (x_1, x_2, \dots, x_n)^T \in R^n$, $x_0 = (x_0^1, x_0^2, \dots, x_0^n)^T \in R^n$, $f = (f_1, f_2, \dots, f_n)^T$, and A is an $n \times n$ diagonal matrix with diagonal elements $a_i, i = 1, 2, \dots, n$.

The basic assumptions throughout the paper are that for each $i = 1, 2, \dots, n$, $f_i : R_+ \times R^n \rightarrow R$ is continuous, and $a_i : R_+ \rightarrow R$ is continuous, where $R_+ = [0, \infty)$.

Our objective is to show the existence of asymptotically stable solutions of the I.V.P. (1) in terms of the following definition found in [1],

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[2], and [6].

Definition 1. A function x is said to be asymptotically stable solution of the I.V.P. (1) if for every $\epsilon > 0$, there exists a $T = T(\epsilon)$ such that for every $t \geq T$, and for every other solution y of (1), $|x(t) - y(t)| \leq \epsilon$.

We assume that the function f does not satisfy a Lipschitz condition in x on a domain $D \subset R_+ \times R^n$ containing the initial point $(0, x_0)$. This will allow the I.V.P. (1) to have more than one solution. As indicated in [6] that it is crucial to have the non-uniqueness of solutions for the kind of stability property that we study in this paper. At the end of the paper, in Corollary 1, we show that the I.V.P. (1) has a unique solution on R_+ if the function f satisfies a Lipschitz condition.

We remark that the above definition of asymptotic stability is different than the standard definition of the Liapunov asymptotic stability. Liapunov asymptotic stability would assume that the I.V.P. has a unique solution x , and then it would ask that the solution x will stay close to another unique solution passing through another initial point that is arbitrarily close to the initial point for x , and that the two solutions would converge. In general, the researchers in this field study various Liapunov stability of the zero solution, assuming that the zero is a solution of the problem. Although the most popular method of study in Liapunov stability is the Liapunov's direct method, it has been found in some recent studies, that for certain problems, fixed point theory can be used very effectively, and that the fixed point theory can remove some of the difficulties that can arise in Liapunov's direct method. For more on this, we refer the interested readers to [4] and [5], and the references therein.

First we convert the above initial value problem into the following equivalent Volterra integral equation.

$$x(t) = u(t) + \int_0^t D(t, s)f(s, x(s))ds, \quad (2)$$

where $u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T$, with $u_i(t) = x_0^i e^{-\int_0^t a_i(s)ds}$, for each $i = 1, 2, \dots, n$, and the kernel D is the $n \times n$ diagonal matrix function with diagonal elements being $d_i(t, s) = e^{-\int_s^t a_i(v)dv}$, $i = 1, 2, \dots, n$. We remark that the continuity of a_i for each i implies the continuity of u for $t \geq 0$, and the continuity of D for $0 \leq s \leq t < \infty$.

We show the existence of asymptotically stable solutions of (1) in Theorem 1, Section 2, by showing the same for the equivalent integral equation (2). In the analysis, we apply the following version of Schauder's fixed point theorem.

Schauder's Fixed Point Theorem. If S is a closed, bounded, convex subset of a Banach space X , and $H : S \rightarrow S$ is completely continuous, then H has a fixed point in S .

An operator is completely continuous if it is continuous and it maps bounded sets into relatively compact sets.

Like Schauder's theorem, a fixed point theorem normally requires a compact mapping. For problems on finite intervals this compactness is usually obtained by Arzela - Ascoli's theorem. In the study of asymptotic stability by fixed point theory, it has been recognized that compactness on infinite interval presents problem. Banaś and Rzepka [2] studied the asymptotic stability employing a fixed point theorem of Darbo type associated with measures of noncompactness. Burton and Zheng [6] avoided the measures of noncompactness and studied the asymptotic stability using Krasnoselskii's theorem, under relatively weaker conditions. They obtained the compactness using a Arzela-Ascoli type argument. Avramescu and Vladimirescu [1] employed Schauder's theorem and obtained the required compactness of the mapping in a different way. We follow the method by Avramescu and Vladimirescu for the compactness of the mapping that we use in this paper.

In addition to our basic assumptions, we also assume that the following conditions hold. For each $i = 1, 2, \dots, n$,

(A1) there exists constant $\bar{a}_i > 0$ such that

$$e^{-\int_0^t a_i(s)ds} \leq \bar{a}_i, \forall t \in R_+,$$

(A2) there exists constant $m_i > 0$ such that

$$\int_0^t d_i(t, s)ds \leq m_i, \forall t \in R_+,$$

(A3) for any $T > 0$,

$$\lim_{t \rightarrow \infty} \int_0^T d_i(t, s)ds = 0,$$

(A4) there exists constant $l_i > 0$ such that

$$\lim_{t \rightarrow \infty} \int_0^t d_i(t, s)ds = l_i,$$

(A5) there exists constant $c_i \geq 0$ such that

$$\lim_{t \rightarrow \infty} e^{-\int_0^t a_i(s)ds} = c_i,$$

As an example, if a_i is a positive constant function for each i , then assumptions (A1)-(A5) are easily satisfied.

We write

$$\int_0^t D(t,s)f(s,x(s))ds = \int_0^t \bar{D}(t,s)\bar{f}(s,x(s))ds, \quad (3)$$

where \bar{D} is the $n \times n$ diagonal matrix function with diagonal elements being $\frac{1}{l_i}e^{-\int_s^t a_i(v)dv}$, $i = 1, 2, \dots, n$, $\bar{f} = (l_1 f_1, l_2 f_2, \dots, l_n f_n)^T$, where l_i is the constant of (A4) for each i .

For any $\rho > 0$, let

$$B_\rho := \{(x \in R^n, |x| \leq \rho)\},$$

where $|\cdot|$ is a vector norm $\in R^n$.

We assume that there exists a $\theta := (\theta_1, \theta_2, \dots, \theta_n)^T \in R^n$, such that $\lim_{t \rightarrow \infty} f_i(t, x) = \theta_i$, $i = 1, 2, \dots, n$, and the limit being uniform with respect to all $x \in B_\rho$. This is same as to assume that

$$(A6) \lim_{t \rightarrow \infty} f(t, x) = \theta,$$

the limit being uniform for all $x \in B_\rho$.

Proposition 1. If assumptions (A2)- (A4), and (A6) hold, then there exists a $\omega \in R^n$ such that

$$\lim_{t \rightarrow \infty} \int_0^t D(t,s)f(s,x(s))ds = \omega,$$

the limit being uniform for all $x \in B_\rho$.

We will use the following lemma in the proof of this proposition. The proof of this lemma is given in [1], and hence we omit it in the present article.

Lemma 1. For a $n \times n$ matrix function K , suppose the following hypotheses hold:

(H1) there exists $M > 0$, such that

$$\int_0^t |K(t,s)|ds \leq M, \forall t \in R_+,$$

(H2) for any $T > 0$, one has

$$\lim_{t \rightarrow \infty} \int_0^T K(t,s)ds = O_{n \times n},$$

(H3)

$$\lim_{t \rightarrow \infty} \int_0^t K(t, s) ds = I_{n \times n}.$$

Then for every $x \in C_l$,

$$\lim_{t \rightarrow \infty} \int_0^t K(t, s)x(s) ds = \lim_{t \rightarrow \infty} x(t).$$

Here C_l denotes the subspace of bounded continuous functions, where each function's limit exists. The precise definition of C_l is given later in the paper. In (H1), $|K|$ denotes the matrix norm on K induced by the vector norm $|\cdot|$ in R^n , and in (H2), $O_{n \times n}$ is the zero matrix. In (H3), $I_{n \times n}$ is the identity matrix.

Proof of Proposition 1. In (3) we find

$$\int_0^t D(t, s)f(s, x(s)) ds = \int_0^t \bar{D}(t, s)\bar{f}(s, x(s)) ds. \quad (4)$$

One can easily verify that \bar{D} satisfies conditions (H1), (H2), and (H3) of Lemma 1. Conditions (H1), (H2), and (H3) follows from assumptions (A2), (A3), and (A4) respectively.

Therefore, from Lemma 1 we obtain

$$\lim_{t \rightarrow \infty} \int_0^t \bar{D}(t, s)\bar{f}(s, x(s)) ds = \omega,$$

where $\omega := (l_1\theta_1, l_2\theta_2, \dots, l_n\theta_n)^T$, where l_i is the constant in (A4) for each $i = 1, 2, \dots, n$. The limit being uniform for all $x \in B_\rho$.

The conclusion of the proof of Proposition 1 now follows from (4).

Let

$$BC := \{x : R_+ \rightarrow R^n, x \text{ bounded and continuous}\}.$$

Then BC is a Banach space with the norm $\|x\| = \sup_{t \geq 0} |x(t)|$, where $|\cdot|$ is a vector norm in R^n .

Define the space $C_l \subset BC$ by

$$C_l := \{x \in BC, \lim_{t \rightarrow \infty} x(t) \in R^n \text{ exists}\}.$$

Definition 2. A family $\mathcal{A} \subset C_l$ is called equiconvergent if for every $\epsilon > 0$, there exists a $T(\epsilon) > 0$, such that for all $x \in \mathcal{A}$, and for all $t_1, t_2 \geq T$, $|x(t_1) - x(t_2)| \leq \epsilon$.

On the space C_l the following compactness criterion holds (see [1]).

Lemma 2. A family $\mathcal{A} \subset C_l$ is relatively compact if and only if

- (a) \mathcal{A} is uniformly bounded,
- (b) \mathcal{A} is equicontinuous on compact subsets of R_+ ,
- (c) \mathcal{A} is equiconvergent .

2. Asymptotically Stable Solutions

For a constant $\rho > 0$, consider the set B_ρ that we defined earlier, i.e., $B_\rho := \{x \in R^n, |x| \leq \rho\}$.

In this section we prove an existence result of asymptotically stable solutions of the I.V.P. (1) for continuous f where $f : R_+ \times B_\rho \rightarrow R^n$.

Define $m_\rho := \sup\{|f(t, x)|, t \in R_+, x \in B_\rho\}$. We assume $m_\rho < \infty$.

Define $S_\rho := \{x \in C_l, \|x\| \leq \rho\}$, Clearly the set S_ρ is a closed, bounded, and convex subset of the Banach space BC .

To prove our main result given in Theorem 1, we assume that

(A7) there exists a function ϕ on R_+ such that

$$\phi(t) = \{\sup|f(t, x) - \theta|, x \in B_\rho\}, t \in R_+,$$

$$\text{with } \lim_{t \rightarrow \infty} \int_0^t |D(t, s)|\phi(s)ds = 0.$$

Once again, if each a_i is a positive constant function, then $D(t, s) = C(t - s)$, a convolution kernel, and assumption (A7) holds due to a known result called the ‘‘convolution lemma’’ which can be found in ([3], p. 74).

Now we present the main result of of this paper, the existence of asymptotically stable solutions of the I.V.P. (1) in the following theorem. Since the I.V.P. (1) is equivalent to (2), we prove the existence of such solutions of (2).

Theorem 1. Suppose assumptions (A1)-(A7) hold. Then equation (2) has at least one solution in S_ρ , and every solution of equation (2) in S_ρ is asymptotically stable.

Proof. By assumption (A2), there exists a constant $M > 0$ such that

$$\int_0^t |D(t, s)|ds \leq M, \forall t \in R_+.$$

From assumption (A1) along with the basic continuity assumption, it follows that the function u_i , for each i , is bounded and continuous. This means there exists a positive constant $\bar{u} \in \mathbb{R}^n$ such that $\sup_{t \geq 0} |u(t)| < \bar{u}$.

Now we choose a $\rho > 0$ such that

$$\bar{u} + Mm_\rho \leq \rho. \quad (5)$$

For $x \in S_\rho$, define H by

$$Hx(t) = u(t) + \int_0^t D(t, s)f(s, x(s))ds. \quad (6)$$

First we show that for $x \in S_\rho$, the function $Hx(t)$ is continuous in t . For any $0 \leq t_1 \leq t_2$, we have

$$\begin{aligned} |Hx(t_1) - Hx(t_2)| &\leq |u(t_1) - u(t_2)| \\ &\quad + m_\rho \int_0^{t_1} |D(t_1, s) - D(t_2, s)|ds \\ &\quad + m_\rho \int_{t_1}^{t_2} |D(t_2, s)|ds. \end{aligned} \quad (7)$$

If $|t_1 - t_2| \rightarrow 0$, then the first term on the right hand side of the above expression $\rightarrow 0$ because $u(t)$ is continuous. Each of the second and third terms $\rightarrow 0$ as $|t_1 - t_2| \rightarrow 0$ because $D(t, s)$ is continuous for $0 \leq s \leq t$. This shows that $Hx(t)$ is continuous in t .

By assumption (A6) we have for all $x \in S_\rho$,

$$\lim_{t \rightarrow \infty} f(t, x) = \theta.$$

Then by Proposition 1, there exists a $\omega \in \mathbb{R}^n$ such that

$$\lim_{t \rightarrow \infty} \int_0^t D(t, s)f(s, x(s))ds = \omega, \quad (8)$$

the limit being uniform with respect to $x \in S_\rho$.

From assumption (A5), there exists a constant vector $\beta = (c_1x_0^1, c_2x_0^2, \dots, c_nx_0^n)^T \in \mathbb{R}^n$ such that

$$\lim_{t \rightarrow \infty} u(t) = \beta. \quad (9)$$

Employing (8) and (9) in (6), we obtain a limit $\kappa := \beta + \omega \in \mathbb{R}^n$, such that

$$\lim_{t \rightarrow \infty} Hx(t) = \kappa, \quad (10)$$

the limit being uniform with respect to $x \in S_\rho$. Thus, we find $HS_\rho \subset C_l$. In addition, from (5) and (6), we obtain

$$\begin{aligned} |Hx(t)| &\leq |u(t)| + \int_0^t |D(t,s)||f(s,x(s))|ds \\ &\leq \rho. \end{aligned}$$

So, $HS_\rho \subset S_\rho$.

Now we prove that H is a continuous operator. For that, let us define operators U and V as follows. For each $x \in S_\rho$,

$$(Ux)(t) = \int_0^t D(t,s)x(s)ds,$$

and

$$(Vx)(t) = f(t,x(t)),$$

for all $t \in R_+$. Clearly, V is continuous in x because f is. The operator U is a linear operator and hence is continuous. The continuity of the operator H is then follows from $Hx = u + (U \circ V)x$, for all $x \in S_\rho$.

Now, we claim that HS_ρ is relatively compact. Since $HS_\rho \subset S_\rho$, the set HS_ρ is uniformly bounded. Also, HS_ρ is equicontinuous on compact subsets of R_+ . For this, it is sufficient to show that HS_ρ is equicontinuous on interval $[0, \gamma]$, for any $\gamma > 0$. Applying the same arguments that we applied to obtain (7), we see that for any $x \in S_\rho$, and for $t_1, t_2 \in [0, \gamma]$,

$$|(Hx)(t_1) - (Hx)(t_2)| \rightarrow 0$$

as $|t_2 - t_1| \rightarrow 0$.

This proves that HS_ρ is equicontinuous on compact subsets of R_+ .

Also, from (10) we see that HS_ρ is equiconvergent. Therefore, by Lemma 2, the set HS_ρ is relatively compact.

Then by Schauder's fixed point theorem, there exists at least one x in S_ρ such that $x = Hx$, showing that (2) has at least one solution in S_ρ .

Finally we show that the solution is asymptotically stable by Definition 1 given above.

Let $x, y \in S_\rho$ be two solutions of (2). Since $x(t) = (Hx)(t)$, $y(t) =$

$(Hy)(t)$ for all $t \in R_+$, we have

$$\begin{aligned} |x(t) - y(t)| &\leq \int_0^t |D(t, s)[f(s, x(s)) - \theta]| ds \\ &\quad + \int_0^t |D(t, s)[f(s, y(s)) - \theta]| ds \\ &\leq 2 \int_0^t |D(t, s)| \phi(s) ds, \end{aligned}$$

where θ and ϕ are from assumptions (A6) and (A7) respectively. Employing assumption (A7) in the above inequality, we obtain $|x(t) - y(t)| \rightarrow 0$ as $t \rightarrow \infty$. Therefore, we have shown that every solution of (2) in S_ρ is asymptotically stable, which concludes the proof of Theorem 1.

An Example:

Consider the initial value problem with the following scalar differential equation and the initial condition:

$$x'(t) + x(t) = \sqrt{|x(t)|} e^{-t}, \quad x(0) = 0.$$

Here $a(t) = 1$, $u(t) = 0$, $D(t, s) = e^{-(t-s)}$, and $f(t, x) = \sqrt{|x|} e^{-t}$. Notice that this f does not satisfy a Lipschitz condition on any domain containing the initial point $(0, 0)$. For a Lipschitz condition to be satisfied, we need to show that there exists a constant $k > 0$ such that $|f(t, x) - f(t, y)| \leq k|x - y|$ for all $(t, x), (t, y) \in D$. One can easily verify, by taking $y = 0$, that no such k exists because when $x \rightarrow 0$, then $k \rightarrow \infty$.

For this f , $\lim_{t \rightarrow \infty} f(t, x) = 0$. Therefore, θ of assumption (A6) is 0. Then $\phi(t)$ of assumption (A7) is $\sqrt{\rho} e^{-t}$. Clearly, $\lim_{t \rightarrow \infty} \phi(t) = 0$, and hence, by the convolution lemma,

$$\lim_{t \rightarrow \infty} \int_0^t |D(t, s)| \phi(s) ds = 0.$$

as assumed in (A7). Also, we have $m_\rho = \sqrt{\rho}$. Then any ρ that satisfies $\sqrt{\rho} \leq \rho$ indeed satisfy (5). So, we can choose $\rho = 1$. Therefore from Theorem 1, we can say that this initial value problem has at least one solution in S_1 , and all solutions in S_1 are asymptotically stable.

Indeed one can verify that $x_1(t) \equiv 0$, and $x_2(t) = e^{-2t}(-1 + e^{t/2})^2$ are two solutions of this I.V.P., and $\lim_{t \rightarrow \infty} |x_1(t) - x_2(t)| = 0$. Also, note that $|x_1(t)| \leq 1$ and $|x_2(t)| \leq 1$ for all $t \in R_+$.

Corollary 1. Suppose assumptions (A1)-(A6) hold. Also, suppose

the function f in (2) satisfies a lipschitz condition, i.e., there exists a constant $k > 0$ such that

$$|f(t, x) - f(t, y)| \leq k|x - y|$$

for all $(t, x), (t, y) \in D \subset R \times R^n$; D contains the point $(0, x_0)$. Then there exists only one solution of (2) in S_ρ .

Proof. From Theorem 1 one can easily see that Schauder's theorem guarantees the existence of at least one solution in S_ρ . To show that there is only one solution, let us assume that there exists two solutions, x and y of (2) in S_ρ . We show, employing Gronwal's inequality, that on any arbitrary interval $[0, T], T > 0$, these solutions are the same i.e., $x(t) = y(t)$ for all $t \in [0, T]$. This concludes that there exists only one solution in S_ρ on R_+ since $T > 0$ is arbitrary. To show $x(t) = y(t)$ for all $t \in [0, T]$, Let $\sigma(t) = |x(t) - y(t)|$. Since f satisfies the above Lipschitz condition, we get from (2) that for $t \in [0, T]$,

$$0 \leq \sigma(t) \leq 0 + \int_0^t k|D(t, s)|\sigma(s)ds, \quad 0 \leq t \leq T.$$

Since $D(t, s)$ is continuous for $0 \leq s \leq t \leq T$, it is bounded by a positive constant, say b . Then ploying the Gronwal's inequality on the above expression we easily obtain

$$0 \leq \sigma(t) \leq 0 \cdot e^{\int_0^t kbds} = 0 \cdot e^{kbT} = 0.$$

This means $\sigma(t) = 0$, i.e., $x(t) = y(t)$ for all $t, 0 \leq t \leq T$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DAYTON, DAYTON, OH 45469-2316 USA

E-mail address: mislam1@udayton.edu