Existence Results for Functional Dynamic Equations with Delay

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Research Interests

- Dynamic Equations on Time Scales
- Existence and Stability theory of Nonlinear Ordinary Differential Equations
- Fractional Differential Equations
- Functional Differential Equations
- Fuzzy Differential Equations
- Set Differential Equations
- Wavelet Methods to BVPs

Introduction

- Brief Review of Time Scales
- Some recent Results on Dynamic Equations on Time Scales
- Delay Dynamic Equations

Time Scales- A brief Review

A time scale $\mathbf{T}$ is a closed subset of $\mathbb{R}$.

The jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by:

$\sigma(t) = inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = sup\{s \in \mathbb{T} : s < t\}$

The point $t \in \mathbb{T}$ is

- left-dense:$\rho(t) = t$
- left-scattered: $\rho(t) < t$
- right-dense:$\sigma(t) = t$
- right-scattered: $\sigma(t) > t$

Time Scales- A brief Review

The graininess function is defined by: $\nu : \mathbb{T} \to [0, \infty)$:
$\nu(t) := t - \rho(t)$, for all $t \in \mathbb{T}$.

The function $f : \mathbb{T} \to \mathbb{R}$ is called a *rd-continuous* provided that it is continuous at each right-dense point and has a left-sided limit at each left dense point. We write $f \in C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}^n)$.

For $\mathbb{T}$ has a left scattered maximum m, the $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the $\Delta-$ derivative of f at a point $t \in \mathbb{T}^\kappa$, denoted by $f^\Delta(t)$ is the number (if it exists) with the property that given $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in U.$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are $\Delta-$ differentiable and $f^{\Delta}(t)$ is rd-continuous is denoted by $C^1_{rd}(\mathbb{T})$.

In [3], the following results were established:
First Derivative Test: Let $f : \mathbb{T} \to \mathbb{R}$ be a function which is $\Delta-$ differentiable on $\mathbb{T}^{\kappa}$. If f has a local maximum at $t_0 \in \mathbb{T}^{\kappa}$ then

1. $f^{\Delta}(\rho(t_0))f^{\Delta}(t_0) \leq 0$ if $t_0 \in \mathbb{T}^{\kappa}$ is not simultaneously ld and rs.
2. $f^{\Delta_l}(t_0)f^{\Delta}(t_0) \leq 0$ if $t_0 \in \mathbb{T}^{\kappa}$ is simultaneously ld and rs and $f \in C^1_{rd}(\mathbb{T})$ where $f^{\Delta_l}(t) = \lim_{s \to t^-} f^{\Delta}(s)$.

Second Derivative Test: If a function $f : \mathbb{T} \to \mathbb{R}$ has a local maximum at a point $t_0 \in \mathbb{T}^{\kappa^2}$ then,

1. $f^{\Delta\Delta}(\rho(t_0)) \leq 0$ provided that t_0 is not simultaneously ld and rs and that $f^{\Delta\Delta}(\rho(t_0))$ exists.
2. $f^{\Delta\Delta_\tau}(t_0) \leq 0$ provided that t_0 is simultaneously ld and rs and that $f \in C^1_{rd}(\mathbb{T})$ where $f^{\Delta\Delta_\tau}(t_0) = f^{\Delta}(t_0) - f^{\Delta_l}(t_0)$.

Consider the IVP: $x^{\Delta}(t) = f(x,t), \quad x(t_0) = x_0, \quad t \in \mathbb{T}^{\kappa}$. where $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ and $t_0 \in \mathbb{T}$.
The following theorem was "proved" in [5]:

Theorem: Let $f \in C_{rd}(\mathbb{R}_0, \mathbb{R})$, where $R_0 = [t_0, t_0 + a]_{\mathbb{T}} \times B, B = \{x \in \mathbb{R} : |x - x_0| \leq b\}$. Then, the above IVP has at least one solution $x(t)$ on $[t_0, t_0 + \alpha]$, $\alpha = min(a, \frac{b}{M})$, and $M = sup_{R_0}|f(x,t)|$.

The following example (see [1]) illustrates that it is not enough to require $f(t,x)$ is rd-continuous function for a.e. $t \in \mathbb{T}$.

Example: Consider the Cauchy problem $x^{\Delta}(t) = f(t, x(t)), \quad x(0) = 0$ on the time scale $\mathbb{T} = \{\frac{1}{n} : n \in \mathbb{N}\} \bigcup \{0\}$, where

$$f(t,x) = \begin{cases} 0, & x \neq 0 \\ t, & x = 0. \end{cases}$$

We have f is rd-continuous since for any continuous function $x : \mathbb{T} \to \mathbb{R}$ the function $t \to f(t, x(t))$ is continuous at any isolated point $t = \frac{1}{n}$ and

$$\lim_{t \to 0} f(t, x(t)) = \lim_{n \to \infty} f(\frac{1}{n}, x(\frac{1}{n})) = 0 = f(0, x(0)).$$

Let x be a solution of the IVP on some interval $[0, a) \cap \mathbb{T}$. Define $x_n = x(\frac{1}{n}), f_n = f(\frac{1}{n}, x_n)$ for $n > \frac{1}{a}$. If $x_n = 0$ for some n, note that

$$f_{n+1} = x^{\Delta}(\frac{1}{n+1}) = \frac{x_n - x_{n+1}}{\frac{1}{n} - \frac{1}{n+1}}.$$

Then we have:

$$f_{n+1} = \begin{cases} 0 & if \;\; x_{n+1} = 0 \\ \neq 0 & if \;\; x_{n+1} \neq 0. \end{cases}$$

This contradicts the definition of f. Thus, $x_n \neq 0$ for all $n > \frac{1}{a}$ and thus $f_n = 0$. So, x_n is a nonzero constant sequence, so $x(0) \neq 0$ from the continuity of x. Thus, the IVP has no solutions.

Delay Differential Equations

Comment: Not all time scales are suitable for study of all types of DDE.

- Given a time scale, one may find a proper delay function. For $x : \mathbb{T} \to \mathbb{R}$, for a given time scale, characterization of $\tau(t)$, the variable delay term such that $t \pm \tau(t) \in \mathbb{T}$ is rarely done. For a very recent detailed classification and analysis of delay terms we refer to [6].
- Alternatively, most authors simply assume that for any $t, s \in \mathbb{T}$, $t + s \in \mathbb{T}$.

In [1], the authors take the following approach:

For $F : \mathbb{T} \times B \to \mathbb{R}$, where B is a Banach space with the norm $\|.\|$, define

$$F^*(t,x) = \begin{cases} F(t,x), & t \in \mathbb{T} \\ F(\tau,x), & t \in (\tau,\sigma(\tau)),\ \tau \in \mathbb{T} \end{cases}$$

Definition: The function F is said to satisfy the Caratheodory condition if

1. $F^*(.,x)$ is a measurable function for every $x \in B$,
2. $F^*(t,.)$ is continuous function for a.e. $t \in \mathbb{T}$,
3. for every $r > 0$ and $t \in \mathbb{T}$ there exists $m \in L^1(0,t)$ such that $|F^*(s,x)| \le m(s)$ for all $s \in (0,t), \|x\| \le r$.

Let $co\mathbb{T}$ denote the convex hull of $\mathbb{T}$. For $x : [-h, 0] \bigcup \mathbb{T} \to \mathbb{R}$, define $\overline{x} : [-h, 0] \bigcup co\mathbb{T} \to \mathbb{R}$ and $x_t : [-h, 0] \to \mathbb{R}$ by

$$\overline{x}(t) = \begin{cases} x(t), & t \in [-h, 0] \bigcup \mathbb{T}, \\ \frac{(\sigma(s)-t)x(s)+(t-s)x(\sigma(s))}{\sigma(s)-s}, & t \in (s, \sigma(s)), \end{cases}$$

$x_t(s) = \overline{x}(t + s), \quad s \in [-h, 0]$.

- If $x^{\Delta}(t)$ exists for some $t \in \mathbb{T}$, it is equal to the RHS derivative of $\overline{x}$ at t.
- x_t is continuous if the function x is continuous.

In [1], the following Cauchy problem is considered:

$$\begin{aligned} x^{\Delta}(t) &= f(t, x(t), \int_0^t g(s, x_s)\Delta s),\ t \in \mathbb{T} \\ x_0 &= \phi. \end{aligned}$$

where $f : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ g : \mathbb{T} \times C([-h, 0], \mathbb{R}) \to \mathbb{R}$, $\phi \in C[-h, 0]$.

A solution of this Cauchy problem is a function $x : [-h, 0] \bigcup [0, a] \to \mathbb{R}$ if $x_0 = \phi$ and

$$x(t) = \phi(0) + \int_0^t f^* \left(\tau, \overline{x}(\tau), \int_0^\tau g^*(s, \overline{x}_{\rho(s)}\ ds \right)\ d\tau.$$

for $t \in \mathbb{T}_{[0,a)}$, where $\overline{x}(s) = \begin{cases} x(s) & s \in \mathbb{T} \\ x(\tau), s \in (\tau, \sigma(\tau)) \end{cases}$

The following existence theorem is established in [1]:

Theorem: Suppose that the functions f and g satisfy the Caratheodory conditions. Then there is a solution of the Cauchy problem on $[-h, a)$ for some $a > 0$.

We now introduce a different approach to study FDE with delay on time scales:

Let $\mathbb{T}$ be a time scale and by $[a, b]$ we mean $\{t \in \mathbb{T} | a \leq t \leq b\}$.

Let $t_0, \tau > 0$ and $\mathbb{T}$ be a time scale and consider $I = [t_0 - \tau, t_0] \subset \mathbb{T}$.

Let $C^n = C[[t_0 - \tau, t_0], \mathbb{R}^n]$ denote the (Banach) space of all continuous functions $\phi : [t_0 - \tau, t_0] \to \mathbb{R}^n$.

For any $\phi \in C^n$, define $\|\phi\| = max_{s \in I} \|\phi(s)\|$.

For each $t \in [t_0, \infty) \bigcap \mathbb{T}$ we define the "transfer function" as follows:

$$\beta_t : [t_0 - \tau, t_0] \to \mathbb{T}$$

$$\beta_t(s) = inf_{s' \in [s, t_0]} \left\{ t - t_0 + s' \in [t_0, \infty) \bigcap \mathbb{T} \right\}$$

- $\beta_t(s)$ is well defined.
- If $\mathbb{T} = \mathbb{R}$, β_t maps the interval $[t_0 - \tau, t_0]$ into $[t - \tau, t]$.
- β_t is monotone and nondecreasing.
- We do not require that $(t - \tau) \in \mathbb{T}$.

Let $x \in C_{rd}[[t_0 - \tau, \infty), \mathbb{R}^n]$.
For any $t \geq t_0$, let x_t denote the translation of the restriction of x to the interval $[t - \tau, t]$.

Let $x_t(s) = x(\beta_t(s))$, for $t_0 - \tau \leq s \leq t_0$.

Let $\rho > 0$ be a given constant and $C_\rho = \{\phi \in C^n | \|\phi\| < \rho\}$.

Consider $x^\Delta(t) = f(t, x_t)$.

A function, $x(t, t_0, \phi_0)$ is said to be a solution of the above FDE with a given initial function $\phi_0 \in C_\rho$, that is, $x(t) = \phi_0(t), \quad t \in [t_0 - \tau, t_0]$ if there exists a number A such that

1. $x(t, t_0, \phi_0)$ is defined and is continuous on $[t - 0 - \tau, t_0 + A]$ and $x_t(t_0, \phi_0) \in C_\rho$ for $t \in [t - 0, t_0 + A]$.
2. $x_{t_0}(t_0, \phi_0) = \phi_0$.
3. $x^\Delta(t, t_0, \phi_0)$ exists for $t \in [t_0, t_0 + A]$ and

$$x^\Delta(t, t_0, \phi_0) = f(t, x_t) \text{ for } t \in [t_0, t_0 + A].$$

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