

12-1-2005

# Disc Separation of the Schur Complement of Diagonally Dominant Matrices and Determinantal Bounds

Jianzhou Liu  
*Xiangtan University*

Fuzhen Zhang  
*Nova Southeastern University, [zhang@nova.edu](mailto:zhang@nova.edu)*

Follow this and additional works at: [https://nsuworks.nova.edu/math\\_facarticles](https://nsuworks.nova.edu/math_facarticles)

 Part of the [Mathematics Commons](#)

---

## NSUWorks Citation

Liu, Jianzhou and Zhang, Fuzhen, "Disc Separation of the Schur Complement of Diagonally Dominant Matrices and Determinantal Bounds" (2005). *Mathematics Faculty Articles*. 51.  
[https://nsuworks.nova.edu/math\\_facarticles/51](https://nsuworks.nova.edu/math_facarticles/51)

This Article is brought to you for free and open access by the Department of Mathematics at NSUWorks. It has been accepted for inclusion in Mathematics Faculty Articles by an authorized administrator of NSUWorks. For more information, please contact [nsuworks@nova.edu](mailto:nsuworks@nova.edu).

## DISC SEPARATION OF THE SCHUR COMPLEMENT OF DIAGONALLY DOMINANT MATRICES AND DETERMINANTAL BOUNDS\*

JIANZHOU LIU<sup>†</sup> AND FUZHEN ZHANG<sup>‡</sup>

**Abstract.** We consider the Geršgorin disc separation from the origin for (doubly) diagonally dominant matrices and their Schur complements, showing that the separation of the Schur complement of a (doubly) diagonally dominant matrix is greater than that of the original grand matrix. As application we discuss the localization of eigenvalues and present some upper and lower bounds for the determinant of diagonally dominant matrices.

**Key words.** Brauer theorem, comparison matrix, diagonally dominant matrix, doubly diagonally dominant matrix, Geršgorin theorem,  $H$ -matrix,  $M$ -matrix, Schur complement, separation

**AMS subject classifications.** 15A45, 15A48

**DOI.** 10.1137/040620369

**1. Introduction.** Let  $A$  be an  $n \times n$  complex matrix. The famous Geršgorin theorem gives a union of discs in the complex plane that contain all eigenvalues of  $A$ . An individual disc comprises the complex numbers  $z$  for which

$$|z - a_{ii}| \leq P_i(A),$$

where

$$P_i(A) = \sum_{j=1, j \neq i}^n |a_{ij}|.$$

This reveals immediately that if  $A$  is *strictly diagonally (row) dominant*, i.e.,

$$|a_{ii}| > P_i(A)$$

for all  $i = 1, 2, \dots, n$ , then no discs contain  $O$ , the origin, and thus  $A$  is nonsingular. In such a case, the quantities  $|a_{ii}| - P_i(A)$  measure the separations of the discs from the origin and give estimates of the “shortest” eigenvalue of the matrix.

A well-known result due to Brauer generalizes the Geršgorin theorem to a union of ovals (of Cassini) that are guaranteed to contain all eigenvalues of  $A$  [7, p. 380]. An oval is given to comprise all complex numbers  $z$  satisfying

$$|z - a_{ii}| |z - a_{jj}| \leq P_i(A) P_j(A), \quad i < j.$$

---

\*Received by the editors December 7, 2004; accepted for publication (in revised form) by H. J. Woerdeman May 18, 2005; published electronically December 7, 2005. This work was supported in part by the Science and Research Fund of Education Department (02A010) and the Natural Science Foundation (04JJ40005) of Hunan Province of China, and also by the FCAS Faculty Development Fund of Nova Southeastern University.

<http://www.siam.org/journals/simax/27-3/62036.html>

<sup>†</sup>Department of Mathematics, Xiangtan University, Xiangtan, Hunan 411105, China (liujz@xtu.edu.cn).

<sup>‡</sup>Division of Math, Science and Technology, Nova Southeastern University, 3301 College Ave., Fort Lauderdale, FL 33314, and School of Mathematics and Systems Science, Shenyang Normal University, Shenyang, Liaoning 110034, China (zhang@nova.edu).

The Geršgorin discs and the Cassini ovals are both effective tools for locating the eigenvalues (spectrum) of a square matrix. As the former ensures the nonsingularity of the strictly diagonally dominant matrices (SD), the latter guarantees the same property of the strictly doubly diagonally dominant matrices (SDD) [7, p. 381]; recall that an *SDD* [10, 13] is a matrix such that for all  $i$  and  $j$ ,  $1 \leq i < j \leq n$ ,

$$(1.1) \quad |a_{ii}||a_{jj}| > P_i(A)P_j(A).$$

Doubly diagonally dominant matrices (for which equality in (1.1) is allowed to occur) are generalizations of diagonally dominant matrices. Both matrix classes possess the properties of nonsingularity and closedness under the Schur complementation (see, e.g., [2, 13]). These properties have been demonstrated to serve as rich and basic tools in numerical analysis (mostly for convergence of iterations) and in matrix analysis (mostly for deriving matrix inequalities). As a result of Kahan (see [15]) shows a relation between the fields of values of a matrix and its Schur complement, this study shows a relation between the Geršgorin discs of a matrix and its Schur complement.

In this paper we first consider the disc separations of the Schur complement; more precisely, we compare a disc separation of the Schur complement to that of the original matrix and show that each Geršgorin disc of the Schur complement is paired with a particular Geršgorin disc of the original matrix; the latter is further from the origin than the former. As applications of our main results, we then discuss localization of eigenvalues and present some upper and lower bounds for the determinants of the diagonally dominant matrices.

We refer the reader to [18, 19] for the *spread* of a Hermitian matrix, which is defined to be the largest eigenvalue of the matrix minus the smallest eigenvalue. It measures the separation between the extreme eigenvalues. One may refer to [1] for general theory on nonnegative matrices and *M*-matrices, [3, 4, 5, 6, 9, 13, 16] for detailed discussions on (generalized) diagonally dominant matrices, [11, 12, 14, 17] for computational aspects of *H*-matrices, and [21] for an extensive survey of the results on the Schur complement.

**2. Lemmas.** We begin this section by recalling a few terms and results that are to be used to prove our theorems. Let  $A = (a_{ij})$  be an  $n \times n$  complex matrix. The *comparison matrix*, denoted by  $\mu(A) = (c_{ij})$ , of  $A$  is defined to be

$$c_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j, \\ -|a_{ij}| & \text{if } i \neq j. \end{cases}$$

A matrix  $A$  is an *M*-matrix if it can be written in the form  $A = mI - P$ , where  $P$  is a nonnegative matrix and  $m > \rho(P)$ , the spectral radius of  $P$ . Note that if  $A$  is an *M*-matrix, then so is the Schur complement of  $A$  and  $\det A > 0$  (see, e.g., [8, Chap. 2, sec. 2.5]). A matrix  $A$  is an *H*-matrix if  $\mu(A)$  is an *M*-matrix. We denote by  $\mathbb{H}_n$  and  $\mathbb{M}_n$  the sets of  $n \times n$  *H*- and *M*-matrices, respectively.

For matrix  $A = (a_{ij})$ , we denote  $|A| = (|a_{ij}|)$ . If the entries of the matrix  $A$  are all nonnegative, then we write  $A \geq 0$ . For real matrices  $A$  and  $B$  of the same size, if  $A - B$  is a nonnegative matrix, we designate  $A \geq B$ .

LEMMA 1. *If  $A$  is an *H*-matrix, then*

$$(2.1) \quad [\mu(A)]^{-1} \geq |A^{-1}|.$$

*Proof.* For the proof, see, e.g., [8, pp. 117, 131].  $\square$

LEMMA 2. *If  $A \in \text{SD}_n$  or  $\text{SDD}_n$ , then  $\mu(A) \in \mathbb{M}_n$ , i.e.,  $A \in \mathbb{H}_n$ .*

LEMMA 3. If  $A \in \text{SD}_n$  or  $\text{SDD}_n$  and  $\alpha$  is a proper subset of  $N \equiv \{1, 2, \dots, n\}$ , then the Schur complement of  $A$  is in  $\text{SD}_{|\alpha^c|}$  or  $\text{SDD}_{|\alpha^c|}$ , where  $\alpha^c = N - \alpha$  is the complement of  $\alpha$  in  $N$  and  $|\alpha^c|$  is the cardinality of  $\alpha^c$ .

Lemmas 2 and 3 can be found in, e.g., [8, p. 114] and [13, Theorem 2.1].

We recall that the Schur complement of  $A$  with respect to the nonsingular submatrix  $A(\alpha)$ , denoted by  $A/A(\alpha)$  or simply  $A/\alpha$ , is defined to be

$$(2.2) \quad A(\alpha^c) - A(\alpha^c, \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha^c),$$

where  $A(\alpha, \beta)$  stands for the submatrix of  $A$  lying in the rows indexed by  $\alpha$  and the columns indexed by  $\beta$ ;  $A(\alpha)$  is an abbreviation for  $A(\alpha, \alpha)$ .

Our next lemma, showing a way of constructing  $M$ -matrices from a given SD matrix, is key to the proofs of our theorems in the following section.

LEMMA 4. Let  $A \in \text{SD}_n$ ,  $\alpha = \{i_1, i_2, \dots, i_k\}$  be a proper subset of  $N$ , and let  $\alpha^c = N - \alpha = \{j_1, j_2, \dots, j_l\}$ ,  $k + l = n$ . For any  $j_t \in \alpha^c$ , denote

$$B_{j_t} \equiv \begin{pmatrix} x & -|a_{j_t i_1}| & \dots & -|a_{j_t i_k}| \\ -\sum_{u=1}^l |a_{i_1 j_u}| & & & \\ \vdots & & \mu[A(\alpha)] & \\ -\sum_{u=1}^l |a_{i_k j_u}| & & & \end{pmatrix}, \quad x > 0.$$

Then  $B_{j_t}$  is a doubly diagonally dominant matrix if and only if

$$(2.3) \quad x \geq \max_{1 \leq w \leq k} \frac{P_{i_w}(A)}{|a_{i_w i_w}|} \sum_{v=1}^k |a_{j_t i_v}|.$$

When strict inequality in (2.3) holds,  $B_{j_t}$  is an  $M$ -matrix of order  $k + 1$ , and thus  $\det B_{j_t} > 0$ . If equality in (2.3) occurs, then  $\det B_{j_t} \geq 0$ .

*Proof.* For simplicity, write  $B_{j_t} \equiv B \equiv (b_{pq})$ . The off-diagonal entries of  $B$  are all nonpositive. Notice that  $b_{11} = x$ ,  $b_{w+1, w+1} = |a_{i_w i_w}|$ ,  $w = 1, 2, \dots, k$ , and

$$P_1(B) = \sum_{v=1}^k |a_{j_t i_v}|, \quad P_{w+1}(B) = P_{i_w}(A).$$

In order for  $B$  to be strictly doubly diagonally dominant, for each  $w$ ,

$$|b_{11}||b_{w+1, w+1}| = x|a_{i_w i_w}| > P_1(B)P_{w+1}(B) = \sum_{v=1}^k |a_{j_t i_v}| P_{i_w}(A),$$

which yields (2.3), while for  $w, y = 1, 2, \dots, k$  with  $w \neq y$ , since  $A \in \text{SD}_n$ ,

$$|b_{w+1, w+1}||b_{y+1, y+1}| = |a_{i_w i_w}||a_{i_y i_y}| > P_{i_w}(A)P_{i_y}(A) = P_{w+1}(B)P_{y+1}(B).$$

Therefore, by definition,  $B$  is strictly doubly diagonally dominant. By Lemma 2,  $B = \mu(B)$  is an  $M$ -matrix and thus  $\det B > 0$ . The equality case follows from a continuity argument (with  $x + \epsilon$  in  $B_{j_t}$  and letting  $\epsilon \rightarrow 0^+$ ).  $\square$

**3. Disc separation of the Schur complement.** Let  $A$  be an  $n \times n$  SD. Then all diagonal entries of  $A$  are necessarily nonzero. Moreover, each disc  $\{z : |z - a_{ii}| \leq P_i(A)\}$  is separated from the origin, and  $|a_{ii}| - P_i(A)$  is the (shortest) distance from the disc to the origin. As the Schur complement of an SD is strictly diagonally dominant, we are interested in the separations of the discs of the Schur complements from the origin compared with those of the original matrix. Our first theorem shows that the discs of the Schur complements are separated further from the origin. For this purpose, consider the ratios  $\frac{|a_{i_v i_v}| - P_{i_v}(A)}{|a_{i_v i_v}|}$ , and define for each  $j_t \in \alpha^c$

$$(3.1) \quad w_{j_t} = \min_{1 \leq v \leq k} \frac{|a_{i_v i_v}| - P_{i_v}(A)}{|a_{i_v i_v}|} \sum_{u=1}^k |a_{j_t i_u}|.$$

Note that for a number  $a$ , nonsingular matrix  $S$ , and vectors  $x$  and  $y$

$$\frac{1}{\det S} \det \begin{pmatrix} a & x \\ y & S \end{pmatrix} = a - xS^{-1}y.$$

We are now ready to present our main result.

**THEOREM 1.** *Let  $A$  be an  $n \times n$  SD,  $\alpha = \{i_1, i_2, \dots, i_k\} \subset N$ ,  $\alpha^c = N - \alpha = \{j_1, j_2, \dots, j_l\}$ ,  $k+l = n$ . Let  $w_{j_t}$  be defined as in (3.1) and denote  $A/\alpha = (a'_{ts})$ . Then*

$$(3.2) \quad |a'_{tt}| - P_t \left( \frac{A}{\alpha} \right) \geq |a_{j_t j_t}| - P_{j_t}(A) + w_{j_t} \geq |a_{j_t j_t}| - P_{j_t}(A) > 0$$

and

$$(3.3) \quad |a'_{tt}| + P_t \left( \frac{A}{\alpha} \right) \leq |a_{j_t j_t}| + P_{j_t}(A) - w_{j_t} \leq |a_{j_t j_t}| + P_{j_t}(A).$$

*Proof.* Note that, by (2.1) and Lemma 2,

$$\{\mu[A(\alpha)]\}^{-1} \geq [A(\alpha)]^{-1}.$$

For  $t = 1, 2, \dots, l$ , by definition of the Schur complement (2.2), we compute

$$\begin{aligned} |a'_{tt}| - P_t \left( \frac{A}{\alpha} \right) &= |a'_{tt}| - \sum_{\substack{s \neq t \\ s=1}}^l |a'_{ts}| \\ &= \left| a_{j_t j_t} - (a_{j_t i_1}, \dots, a_{j_t i_k}) [A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\ &\quad - \sum_{\substack{s \neq t \\ s=1}}^l \left| a_{j_t j_s} - (a_{j_t i_1}, \dots, a_{j_t i_k}) [A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right| \\ &\geq |a_{j_t j_t}| - \sum_{\substack{s \neq t \\ s=1}}^l |a_{j_t j_s}| - \sum_{s=1}^l \left| (a_{j_t i_1}, \dots, a_{j_t i_k}) [A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right| \\ &\geq |a_{j_t j_t}| - \sum_{\substack{s \neq t \\ s=1}}^l |a_{j_t j_s}| - \sum_{s=1}^l (|a_{j_t i_1}|, \dots, |a_{j_t i_k}|) \{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} |a_{i_1 j_s}| \\ \vdots \\ |a_{i_k j_s}| \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 &= |a_{j_t j_t}| - P_{j_t}(A) + \sum_{u=1}^k |a_{j_t i_u}| + w_{j_t} - w_{j_t} \\
 &\quad - \sum_{s=1}^l (|a_{j_t i_1}|, \dots, |a_{j_t i_k}|) \{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} |a_{i_1 j_s}| \\ \vdots \\ |a_{i_k j_s}| \end{pmatrix} \\
 &= |a_{j_t j_t}| - P_{j_t}(A) + w_{j_t} \\
 &\quad + \sum_{u=1}^k |a_{j_t i_u}| - w_{j_t} - (|a_{j_t i_1}|, \dots, |a_{j_t i_k}|) \{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} \sum_{s=1}^l |a_{i_1 j_s}| \\ \vdots \\ \sum_{s=1}^l |a_{i_k j_s}| \end{pmatrix} \\
 &= |a_{j_t j_t}| - P_{j_t}(A) + w_{j_t} \\
 &\quad + \frac{1}{\det\{\mu[A(\alpha)]\}} \det \begin{pmatrix} \sum_{u=1}^k |a_{j_t i_u}| - w_{j_t} & -|a_{j_t i_1}| & \dots & -|a_{j_t i_k}| \\ -\sum_{s=1}^l |a_{i_1 j_s}| & & & \\ \vdots & & \mu[A(\alpha)] & \\ -\sum_{s=1}^l |a_{i_k j_s}| & & & \end{pmatrix} \\
 &= |a_{j_t j_t}| - P_{j_t}(A) + w_{j_t} + \frac{1}{\det\{\mu[A(\alpha)]\}} \det B,
 \end{aligned}$$

where  $B$  is the matrix containing  $\mu[A(\alpha)]$  on the right-hand side. By taking  $x$  in Lemma 4 to be  $\sum_{u=1}^k |a_{j_t i_u}| - w_{j_t}$  (which is greater than or equal to the expression on the right-hand side of (2.3) by computation), we see  $\det B \geq 0$ , and thus (3.2) follows.

In a similar way, one may show that

$$\begin{aligned}
 |a'_{tt}| + P_t \left( \frac{A}{\alpha} \right) &= |a'_{tt}| + \sum_{\substack{s \neq t \\ s=1}}^l |a'_{ts}| \\
 &\leq |a_{j_t j_t}| + P_{j_t}(A) - w_{j_t} - \frac{1}{\det\{\mu[A(\alpha)]\}} \det B \\
 &\leq |a_{j_t j_t}| + P_{j_t}(A) - w_{j_t} \\
 &\leq |a_{j_t j_t}| + P_{j_t}(A).
 \end{aligned}$$

This reveals (3.3).  $\square$

The inequalities (3.2) immediately yield the following well-known result [2].

COROLLARY 1. *The Schur complement of an SD is strictly diagonally dominant.*

COROLLARY 2. *Let  $A \in \text{SD}_n$  and take  $\alpha = \{1, 2, \dots, n - 1\}$ . Then*

$$|a_{nn}| + \max_{1 \leq i \leq n-1} \frac{P_i(A)}{|a_{ii}|} P_n(A) \geq \left| \frac{A}{\alpha} \right| \geq |a_{nn}| - \max_{1 \leq i \leq n-1} \frac{P_i(A)}{|a_{ii}|} P_n(A) > 0.$$

*Proof.* Notice that  $\alpha^c$  contains only one element  $j_t = n$ . Thus,  $A/\alpha$  is nothing

but a number, and so  $P_t(A/\alpha) = 0$ , and, from (3.1),

$$\begin{aligned} w_{j_t} = w_n &= \min_{1 \leq i \leq n-1} \frac{|a_{ii}| - P_i(A)}{|a_{ii}|} \sum_{u=1}^{n-1} |a_{nu}| \\ &= \min_{1 \leq i \leq n-1} \left( 1 - \frac{P_i(A)}{|a_{ii}|} \right) P_n(A) \\ &= P_n(A) - \max_{1 \leq i \leq n-1} \frac{P_i(A)}{|a_{ii}|} P_n(A). \end{aligned}$$

Substituting this into (3.2) and (3.3) results in the desired inequalities.  $\square$

*Remark 1.* Corollary 2 can be proven directly by computation. Note that

$$P_n(A) - w_n = P_n(A) \max_{1 \leq i \leq n-1} \frac{P_i(A)}{|a_{ii}|}.$$

**COROLLARY 3.** *Let  $A = (a_{ij}) \in \text{SD}_n$  (complex). Denote  $A^{-1} = (a^{ij})$ . Then*

$$a_{ii} > 0 \iff a^{ii} > 0.$$

If  $d_+(X)$  stands for the number of positive entries on the main diagonal of  $X$ ,

$$d_+[A(\alpha)] = d_+[A^{-1}(\alpha)].$$

*Proof.* Without loss of generality, take  $a_{ii} = a_{nn}$ . Then  $a^{nn} = (A/\beta)^{-1}$ , where  $\beta = \{1, 2, \dots, n-1\}$  (see, e.g., [20, p. 37]). Since  $A$  is an SD, by (3.2),  $A/\beta > 0$ , and thus  $a^{nn} > 0$ . The identity  $d_+[A(\alpha)] = d_+[A^{-1}(\alpha)]$  follows at once.  $\square$

*Remark 2.* The  $t$ th disc of  $A/\alpha$  is paired with the  $j_t$ th disc of  $A$ .

Now we turn our attention to doubly diagonally dominant matrices. If  $A$  is in SDD but not in SD, by (1.1), there is one and only one index  $i_0$ , say, such that

$$(3.4) \quad |a_{i_0 i_0}| \leq P_{i_0}(A).$$

**THEOREM 2.** *Let  $A$  be an  $n \times n$  SDD and  $i_0, 1 \leq i_0 \leq n$ , be such as in (3.4). Then for any index set  $\alpha$  containing  $i_0$ , writing  $\alpha = \{i_1, i_2, \dots, i_k\}$ ,  $\alpha^c = N - \alpha = \{j_1, j_2, \dots, j_l\}$ , and  $A/\alpha = (a'_{ts})$ ,*

$$(3.5) \quad |a'_{tt}| - P_t \left( \frac{A}{\alpha} \right) \geq |a_{j_t j_t}| - P_{j_t}(A) + \left( 1 - \frac{P_{i_0}(A)}{|a_{i_0 i_0}|} \right) \sum_{v=1}^k |a_{j_t i_v}|$$

$$(3.6) \quad \geq |a_{j_t j_t}| - \frac{P_{i_0}(A)}{|a_{i_0 i_0}|} P_{j_t}(A) > 0$$

and

$$\begin{aligned} |a'_{tt}| + P_t \left( \frac{A}{\alpha} \right) &\leq |a_{j_t j_t}| + P_{j_t}(A) - \left( 1 - \frac{P_{i_0}(A)}{|a_{i_0 i_0}|} \right) \sum_{v=1}^k |a_{j_t i_v}| \\ &\leq |a_{j_t j_t}| + \frac{P_{i_0}(A)}{|a_{i_0 i_0}|} P_{j_t}(A). \end{aligned}$$

*Proof.* Once again we compute, by definition (2.2),

$$\begin{aligned}
 |a'_{tt}| - P_t\left(\frac{A}{\alpha}\right) &= |a'_{tt}| - \sum_{\substack{s \neq t \\ s=1}}^l |a'_{ts}| \\
 &= \left| a_{j_t j_t} - (a_{j_t i_1}, \dots, a_{j_t i_k}) [A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\
 &\quad - \sum_{\substack{s \neq t \\ s=1}}^l \left| a_{j_t j_s} - (a_{j_t i_1}, \dots, a_{j_t i_k}) [A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right| \\
 &\geq |a_{j_t j_t}| - P_{j_t}(A) + \left(1 - \frac{P_{i_0}(A)}{|a_{i_0 i_0}|}\right) \sum_{v=1}^k |a_{j_t i_v}| + \frac{P_{i_0}(A)}{|a_{i_0 i_0}|} \sum_{v=1}^k |a_{j_t i_v}| \\
 &\quad - \sum_{s=1}^l (|a_{j_t i_1}|, \dots, |a_{j_t i_k}|) \{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} |a_{i_1 j_s}| \\ \vdots \\ |a_{i_k j_s}| \end{pmatrix} \\
 &= |a_{j_t j_t}| - P_{j_t}(A) + \left(1 - \frac{P_{i_0}(A)}{|a_{i_0 i_0}|}\right) \sum_{v=1}^k |a_{j_t i_v}| \\
 &\quad + \frac{1}{\det\{\mu[A(\alpha)]\}} \det \begin{pmatrix} \frac{P_{i_0}(A)}{|a_{i_0 i_0}|} \sum_{v=1}^k |a_{j_t i_v}| & -|a_{j_t i_1}| & \dots & -|a_{j_t i_k}| \\ -\sum_{s=1}^l |a_{i_1 j_s}| & & & \\ \vdots & & \mu[A(\alpha)] & \\ -\sum_{s=1}^l |a_{i_k j_s}| & & & \end{pmatrix} \\
 &= |a_{j_t j_t}| - P_{j_t}(A) + \left(1 - \frac{P_{i_0}(A)}{|a_{i_0 i_0}|}\right) \sum_{v=1}^k |a_{j_t i_v}| + \frac{1}{\det\{\mu[A(\alpha)]\}} \det \tilde{B},
 \end{aligned}$$

where  $\tilde{B}$  is the large size matrix containing  $\mu[A(\alpha)]$ . By taking  $x$  in Lemma 4 to be  $\frac{P_{i_0}(A)}{|a_{i_0 i_0}|} \sum_{v=1}^k |a_{j_t i_v}|$ , we see that  $\det \tilde{B} \geq 0$ , and thus (3.5) follows. Replacing  $\sum_{v=1}^k |a_{j_t i_v}|$  in (3.5) by  $P_{j_t}(A)$  results in (3.6). The second set of inequalities in the theorem are similarly proven.  $\square$

**4. Applications: Bounds for determinants and localization of eigenvalues.** In this section we make use of the results in the previous section to present some upper and lower bounds for determinants. Also, as an application of Theorem 1, we show a result on the localization of eigenvalues of the diagonally dominant matrices.

Let  $\{j_1, j_2, \dots, j_n\}$  be a rearrangement of the elements in  $N = \{1, 2, \dots, n\}$ . Denote  $\alpha_1 = \{j_n\}$ ,  $\alpha_2 = \{j_n, j_{n-1}\}$ ,  $\dots$ ,  $\alpha_n = \{j_1, j_2, \dots, j_n\} = N$ . Then  $\alpha_{n-k+1} - \alpha_{n-k} = \{j_k\}$ ,  $k = 1, 2, \dots, n$ , with  $\alpha_0 = \emptyset$ , and

$$P_{j_k}[A(\alpha_{n-k+1})] = \sum_{u \in \alpha_{n-k}} |a_{j_k u}|.$$



Let  $\mathcal{J}$  represent any rearrangement  $\{j_1, j_2, \dots, j_n\}$  of the elements in  $N$  with  $\alpha_1, \alpha_2, \dots, \alpha_n$  defined as above.

**THEOREM 3.** *Let  $A$  be an  $n \times n$  SD. Then*

$$(4.1) \quad \det A \geq \max_{\mathcal{J}} \prod_{k=1}^n \left\{ |a_{j_k j_k}| - \max_{u \in \alpha_{n-k}} \frac{P_u[A(\alpha_{n-k+1})]}{|a_{uu}|} P_{j_k}[A(\alpha_{n-k+1})] \right\}$$

and

$$(4.2) \quad \det A \leq \min_{\mathcal{J}} \prod_{k=1}^n \left\{ |a_{j_k j_k}| + \max_{u \in \alpha_{n-k}} \frac{P_u[A(\alpha_{n-k+1})]}{|a_{uu}|} P_{j_k}[A(\alpha_{n-k+1})] \right\}.$$

*Proof.* For the first inequality, since  $\alpha_{n-k}$  is contained in  $\alpha_{n-k+1}$  and  $\alpha_{n-k+1} - \alpha_{n-k} = \{j_k\}$ , we have, by Corollary 2, for each  $k$ ,

$$\left| \frac{A(\alpha_{n-k+1})}{\alpha_{n-k}} \right| \geq |a_{j_k j_k}| - \max_{u \in \alpha_{n-k}} \frac{P_u[A(\alpha_{n-k+1})]}{|a_{uu}|} P_{j_k}[A(\alpha_{n-k+1})] > 0.$$

It follows that

$$\begin{aligned} \det A &= \left| \frac{\det A}{\det[A(\alpha_{n-1})]} \right| \left| \frac{\det[A(\alpha_{n-1})]}{\det[A(\alpha_{n-2})]} \right| \cdots \left| \frac{\det[A(\alpha_2)]}{\det[A(\alpha_1)]} \right| |\det[A(\alpha_1)]| \\ &= \left| \det \left( \frac{A}{\alpha_{n-1}} \right) \right| \left| \det \left[ \frac{A(\alpha_{n-1})}{\alpha_{n-2}} \right] \right| \cdots \left| \det \left[ \frac{A(\alpha_2)}{\alpha_1} \right] \right| |\det A(\alpha_1)| \\ &= |a_{j_n j_n}| \prod_{k=1}^{n-1} \left| \frac{A(\alpha_{n-k+1})}{\alpha_{n-k}} \right| \\ &\geq |a_{j_n j_n}| \prod_{k=1}^{n-1} \left\{ |a_{j_k j_k}| - \max_{u \in \alpha_{n-k}} \frac{P_u[A(\alpha_{n-k+1})]}{|a_{uu}|} P_{j_k}[A(\alpha_{n-k+1})] \right\}. \end{aligned}$$

This implies (4.1). The inequality (4.2) in the theorem is similarly proven. □

Notice that when  $A \in \text{SD}_n$ , for any  $1 \leq k \leq n$  and  $u \in \alpha_{n-k}$ ,

$$0 \leq \frac{P_u[A(\alpha_{n-k+1})]}{|a_{uu}|} \leq 1.$$

We have the following claim immediately from the theorem.

**COROLLARY 4.** *Let  $A$  be an  $n \times n$  SD. Then*

$$\det A \geq \max_{\mathcal{J}} \prod_{k=1}^n \left\{ |a_{j_k j_k}| - \sum_{u \in \alpha_{n-k}} |a_{j_k u}| \right\}$$

and

$$\det A \leq \min_{\mathcal{J}} \prod_{k=1}^n \left\{ |a_{j_k j_k}| + \sum_{u \in \alpha_{n-k}} |a_{j_k u}| \right\}.$$

*Example.* Let  $n = 3$  and take  $j_1 = 3, j_2 = 1$ , and  $j_3 = 2$ . With  $\alpha_1 = \{j_3\} = \{2\}$ ,  $\alpha_2 = \{j_3, j_2\} = \{1, 2\}$ , and  $\alpha_3 = \{j_3, j_2, j_1\} = \{1, 2, 3\}$ , by Corollary 4, we have for any  $A \in \text{SD}_3$

$$|\det A| \geq |a_{22}|(|a_{33}| - |a_{31}| - |a_{32}|)(|a_{11}| - |a_{12}|)$$

and

$$|\det A| \leq |a_{22}|(|a_{33}| + |a_{31}| + |a_{32}|)(|a_{11}| + |a_{12}|).$$

For an analogous result for SDDs, let  $\mathcal{I}$  denote all rearrangements of the elements in  $N$  with  $\alpha_1 = \{i_0 \equiv j_n\}$ , where

$$|a_{i_0 i_0}| \leq P_{i_0}(A).$$

The proof of the following theorem is similar to that of the previous theorem.

**THEOREM 4.** *Let  $A \in \text{SDD}_n$  with  $|a_{i_0 i_0}| \leq P_{i_0}(A)$ . Then*

$$\det A \geq \max_{\mathcal{I}} |a_{i_0 i_0}| \prod_{k=1}^{n-1} \left\{ |a_{j_k j_k}| - \frac{P_{i_0}[A(\alpha_{n-k+1})]}{|a_{i_0 i_0}|} P_{j_k}[A(\alpha_{n-k+1})] \right\}$$

and

$$\det A \leq \min_{\mathcal{I}} |a_{i_0 i_0}| \prod_{k=1}^{n-1} \left\{ |a_{j_k j_k}| + \frac{P_{i_0}[A(\alpha_{n-k+1})]}{|a_{i_0 i_0}|} P_{j_k}[A(\alpha_{n-k+1})] \right\}.$$

We conclude the article by presenting a result that falls in the localization of eigenvalues of the diagonally dominant matrices.

**THEOREM 5.** *Let  $A \in \text{SD}_n$  be a matrix with real diagonal entries and  $\alpha$  be a proper subset of  $N = \{1, 2, \dots, n\}$ . Then  $A/\alpha$  and  $A(\alpha^c)$  have the same number of eigenvalues whose real parts are greater (less) than  $w$  (resp.,  $-w$ ), where*

$$w = \min_{j \in \alpha^c} \left[ |a_{jj}| - P_j(A) + \min_{i \in \alpha} \frac{|a_{ii}| - P_i(A)}{|a_{ii}|} \sum_{i \in \alpha} |a_{ji}| \right].$$

*Proof.* By (3.2) in Theorem 1, the matrix  $\mu(A/\alpha) - wI$  is diagonally dominant, and thus is  $A/\alpha - wI$ . In addition, by Theorem 1 again, since  $a'_{tt} - w > 0$  if and only if  $a'_{tt} > 0$ , there are the same number of positive entries on the main diagonals of  $A/\alpha - wI$  and  $A/\alpha$ ; that is,  $d_+(A/\alpha - wI) = d_+(A/\alpha)$ . Thus, by the Geršgorin theorem,  $A/\alpha - wI$  has  $d_+(A/\alpha)$  eigenvalues with positive real part (on the open right-half complex plane). On the other hand, the eigenvalues of  $A/\alpha - wI$  are the eigenvalues of  $A/\alpha$  minus  $w$ , so  $A/\alpha$  has  $d_+(A/\alpha)$  eigenvalues with positive real part greater than  $w$ . Observe that  $A^{-1}(\alpha^c) = (A/\alpha)^{-1}$  (see, e.g., [20, p. 184]) and further that  $(A/\alpha)^{-1}$  and  $A/\alpha$  have the same number of eigenvalues with positive real part. By Corollary 3, we conclude that  $A/\alpha$  and  $A(\alpha^c)$  have the same number of eigenvalues whose real parts are greater than  $w$ . For the number of negative parts of eigenvalues, the above argument works with  $-A/\alpha$  in place of  $A/\alpha$ .  $\square$

An immediate geometric explanation of Theorem 5 is that if the principal submatrix  $A(\alpha^c)$  has no eigenvalue between the vertical lines  $x = -w$  and  $x = w$ , i.e., the band  $|z| \leq w$ , then the Schur complement  $A/\alpha$  has no eigenvalue in the band.

REFERENCES

[1] A. BERMAN AND R. J. PLEMMONS, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.  
 [2] D. CARLSON AND T. MARKHAM, *Schur complements of diagonally dominant matrices*, Czech. Math. J., 29 (1979), pp. 246–251.

- [3] M. FIEDLER AND V. PTÁK, *On the matrices with non-positive off-diagonal elements and positive principal minors*, Czech. Math. J., 12 (1962), pp. 382–400.
- [4] M. FIEDLER AND V. PTÁK, *Diagonally dominant matrices*, Czech. Math. J., 17 (1967), pp. 420–433.
- [5] Y.-M. GAO AND X.-H. WANG, *Criteria for generalized diagonally dominant matrices and  $M$ -matrices*, Linear Algebra Appl., 169 (1992), pp. 257–268.
- [6] Y.-M. GAO AND X.-H. WANG, *Criteria for generalized diagonally dominant matrices and  $M$ -matrices II*, Linear Algebra Appl., 248 (1996), pp. 339–353.
- [7] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, New York, 1985.
- [8] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, New York, 1991.
- [9] T.-Z. HUANG, *A note on generalized diagonally dominant matrices*, Linear Algebra Appl., 225 (1995), pp. 237–242.
- [10] K. D. IKRAMOV, *Invariance of the Brauer diagonal dominance in Gaussian elimination*, Moscow Univ. Comput. Math. Cybernet., N2 (1989), pp. 91–94.
- [11] T. KOHNO, H. NIKI, H. SAWAMI, AND Y.-M. GAO, *An iterative test for  $H$ -matrix*, J. Comput. Appl. Math., 115 (2000), pp. 349–355.
- [12] B. LI, L. LI, M. MARADA, H. NIKI, AND M. J. TSATSOMEROS, *An iterative criterion for  $H$ -matrices*, Linear Algebra Appl., 271 (1998), pp. 179–190.
- [13] B. LI AND M. TSATSOMEROS, *Doubly diagonally dominant matrices*, Linear Algebra Appl., 261 (1997), pp. 221–235.
- [14] L. LI, H. NIKI, AND M. SASANABE, *A nonparameter criterion for generalized diagonally dominant matrices*, Int. J. Comput. Math., 71 (1999), pp. 267–275.
- [15] R.-C. LI, *Relations between the Field of Values of a Matrix and Those of Its Schur Complements*, Technical Report UCB/CSD-94-849, EECS Department, University of California, Berkeley, CA, 1994; available online at <http://www.eecs.berkeley.edu/Pubs/TechRpts/1994/5874.html>.
- [16] J. LIU, Y. HUANG, AND F. ZHANG, *The Schur complements of generalized doubly diagonally dominant matrices*, Linear Algebra Appl., 378 (2004), pp. 231–244.
- [17] K. OJIRO, H. NIKI, AND M. USUI, *A new criterion for the  $H$ -matrix property*, J. Comput. Appl. Math., 150 (2003), pp. 293–302.
- [18] L. MIRSKY, *The spread of a matrix*, Mathematika, 3 (1956), pp. 127–130.
- [19] R. C. THOMPSON, *Root spreads for polynomials and Hermitian matrix pencils*, Linear Algebra Appl., 220 (1995), pp. 419–433.
- [20] F.-Z. ZHANG, *Matrix Theory: Basic Results and Techniques*, Springer, New York, 1999.
- [21] F.-Z. ZHANG, ED., *The Schur Complement and Its Applications*, Springer-Verlag, New York, 2005.

Copyright of *SIAM Journal on Matrix Analysis & Applications* is the property of Society for Industrial and Applied Mathematics and its content may not be copied or emailed to multiple sites or posted to a listserv without the copyright holder's express written permission. However, users may print, download, or email articles for individual use.