Fractional Calculus and Smallest Eigenvalues

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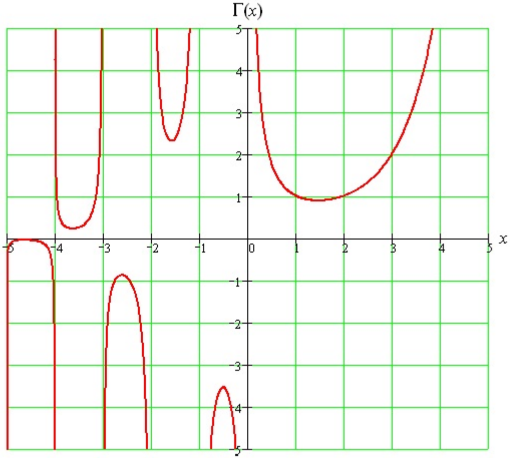
Here are the important properties of the Gamma function.

1. For each $x \in (0, \infty)$, $\Gamma(x+1) = x\Gamma(x)$.
2. For $n \in \mathbb{N}$, $\Gamma(n+1) = n!$.


Γ(x)
x
-5
-4
-3
-2
-1
0
1
2
3
4
5

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In 1695, after Leibniz first invented the notation $\dfrac{d^n y}{dx^n}$, L'Hôpital wrote to Leibniz and asked him, "What if $n = 1/2$?" Leibniz responded, "It leads to a paradox, from which one day useful consequences will be drawn."

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$$\begin{aligned}\frac{d^n}{dx^n}(x^m) &= m\cdot(m-1)\cdot(m-2)\cdots(m-(n+1))x^{m-n}\\ &= \frac{m!}{(m-n)!}x^{m-n},\ n\in\mathbb{N},\ m\geq n.\end{aligned}$$

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He then generalized the formula for all n, obtaining

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He then generalized the formula for all n, obtaining

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Using this definition, he was able to show $\frac{d^{1/2}}{dx^{1/2}}(x) = \frac{2\sqrt{x}}{\sqrt{\pi}}$.

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Fractional calculus finds use in many fields of science and engineering, including fluid flow, electrical networks, and probability.

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$$I_{c^+}^n f(x) = \int_c^x \int_c^{x_1} \cdots \int_c^{x_{n-1}} f(t)dt\, dx_{n-1} \cdots dx_2\, dx_1$$
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For $n = 2$,

$$I_{c^+}^2 f(x) = \int_c^x \int_c^{x_1} f(t)dt\ dx_1$$
$$= \int_c^x \int_t^x f(t)dx_1\ dt$$
$$= \int_c^x (x-t)f(t)dt$$

For $n = 3$, if we use the previous result,

$$\begin{aligned} I_{c^+}^3 f(x) &= \int_c^x \int_c^{x_1} \int_c^{x_2} f(t)dt\, dx_2\, dx_1 \\ &= \int_c^x \left[\int_c^{x_1} \int_c^{x_2} f(t)dt\, dx_2 \right] dx_1 \\ &= \int_c^x \left[\int^{x_1} (x_1 - t) f(t) dt \right] dx_1 \\ &= \int_c^x \int_t^x (x_1 - t) f(t) dx_1\, dt \\ &= \int_c^x f(t) \frac{(x-t)^2}{2} dt. \end{aligned}$$

If we continue in this fashion, we obtain

$$\begin{aligned} I^n_{c^+} f(x) &= \frac{1}{(n-1)!} \int_c^x (x-t)^{n-1} f(t)dt \\ &= \frac{1}{\Gamma(n)} \int_c^x (x-t)^{n-1} f(t)dt. \end{aligned}$$

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Definition

For a function $f(x)$ defined on (c,∞), define the Riemann-Liouville fractional integral of order $\alpha > 0$ of $f(x)$ by

$$I^\alpha_{c^+}f(x)\frac{1}{\Gamma(\alpha)}\int_c^x (x-t)^{\alpha-1}f(t)dt,$$

provided the integral exists.

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Definition

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$$\begin{aligned} D_{c^+}^{\alpha} f(x) &= \frac{d^n}{dx^n} I_{c^+}^{n-\alpha} f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_c^x (x-t)^{n-\alpha-1} f(t) dt, \end{aligned}$$

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$$\begin{aligned} D^{1/2}_{0^+} e^x &= D^{1/2}_{0^+}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \\ &= \frac{\Gamma(1)}{\Gamma(1/2)} x^{-1/2} + \frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2} + \frac{\Gamma(3)}{2!\Gamma(5/2)} x^{3/2} + \frac{\Gamma(4)}{3!\Gamma(7/2)} x^{5/2} + \cdots \\ &= \frac{1}{\sqrt{\pi x}}\left(1 + 2x + \frac{4}{3}x^2 + \frac{8}{15}x^3 + \cdots\right) \\ &\neq e^x. \end{aligned}$$

But for $c = -\infty$, we have

$$
\begin{aligned}
I^{1/2}_{-\infty} e^x &= \frac{1}{\Gamma(1/2)} \int_{-\infty}^{x} (x-t)^{-1/2} e^t \, dt \\
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In general, for $c = -\infty$, $D_{-\infty}^{\alpha} e^{ax} = a^{\alpha} e^{ax}$.

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Similarly, for $g(x) = \cos x$, $g^{(n)}(x) = \cos(x + (n\pi)/2)$. In general,
$D^{\alpha}_{-\infty} \sin x = \sin(x + (\alpha\pi)/2)$ and $D^{\alpha}_{-\infty} \cos x = \cos(x + (\alpha\pi)/2)$.

We consider the comparison of smallest eigenvalues for the eigenvalue problems

$$D_{0+}^{\alpha}u + \lambda_1 p(t)u = 0, \quad 0 < t < 1, \tag{1}$$

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where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville derivative, and $p(t)$ and $q(t)$ are continuous nonnegative functions on $[0,1]$, where neither $p(t)$ nor $q(t)$ vanishes identically on any compact subinterval of $[0,1]$.

Definition

Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided

(i) $\alpha u + \beta v \in \mathcal{P}$, for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and

(ii) $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ implies $u = 0$.

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A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^\circ$, of $\mathcal{P}$, is nonempty. A cone $\mathcal{P}$ is reproducing if $\mathcal{B} = \mathcal{P} - \mathcal{P}$; i.e., given $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$.

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Krasnosel'skii showed that every solid cone is reproducing.

Definition

Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u, v \in \mathcal{B}$, $u \leq v$ with respect to $\mathcal{P}$ if $v - u \in \mathcal{P}$. If both $M, N : \mathcal{B} \to \mathcal{B}$ are bounded linear operators, $M \leq N$ with respect to $\mathcal{P}$ if $Mu \leq Nu$ for all $u \in \mathcal{P}$.

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Definition

A bounded linear operator $M : \mathcal{B} \to \mathcal{B}$ is u_0-positive with respect to $\mathcal{P}$ if there exists $0 \neq u_0 \in \mathcal{P}$ such that for each $0 \neq u \in \mathcal{P}$, there exist $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \leq Mu \leq k_2 u_0$ with respect to $\mathcal{P}$.

Lemma

Let $\mathcal{B}$ be a Banach space over the reals, and let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $M : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $M : \mathcal{P} \backslash \{0\} \to \mathcal{P}^\circ$, then M is u_0-positive with respect to $\mathcal{P}$.

Theorem (Krasnosel'skii)

Let $\mathcal{B}$ be a real Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $L : \mathcal{B} \to \mathcal{B}$ be a compact, u_0-positive, linear operator. Then L has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.

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Theorem (Krasnosel'skii)

Let $\mathcal{B}$ be a real Banach space and $\mathcal{P} \subset \mathcal{B}$ be a cone. Let both $M, N : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators and assume that at least one of the operators is u_0-positive. If $M \leq N$, $My_1 \geq \lambda_1 y_1$ for some $y_1 \in \mathcal{P}$ and some $\lambda_1 > 0$, and $Ny_2 \leq \lambda_2 y_2$ for some $y_2 \in \mathcal{P}$ and some $\lambda_2 > 0$, then $\lambda_1 \leq \lambda_2$. Futhermore, $\lambda_1 = \lambda_2$ implies y_1 is a scalar multiple of y_2.

We derive comparison results for these eigenvalue problems by applying the previous theorems mentioned. To do this, we will define integral operators whose kernel is the Green's function of $-D_{0+}^{\alpha} u(t) = 0$, (3), which is given by

$$G(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{4}$$

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$$\mathcal{P} = \{u \in B \mid u(t) \geq 0 \text{ for } t \in [0,1]\}.$$

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Proof: Define

$$\Omega := \{u \in B \mid u(t) > 0 \text{ for } t \in (0,1), v(0) > 0, v'(1) < 0\}. \quad (5)$$

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$\Omega \subset \mathcal{P}^\circ$. So $\mathcal{P}$ is solid and hence reproducing.

Define the compact linear operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mu(t) = \int_0^1 G(t,s)p(s)u(s)ds \tag{6}$$

and

$$Nu(t) = \int_0^1 G(t,s)q(s)u(s)\, ds. \tag{7}$$

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Lemma

The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.

Proof: We show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(t) \geq 0$. Then since $G(t,s) \geq 0$ on $[0,1] \times [0,1]$ and $p(t) \geq 0$ on $[0,1]$,

$$Mu(t) = \int_0^1 G(t,s)p(s)u(s)ds \geq 0,$$

for $0 \leq t \leq 1$. So $M : \mathcal{P} \to \mathcal{P}$.

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$$\begin{aligned} Mu(t) &= \int_0^1 G(t,s)p(s)u(s)ds \\ &\geq \int_\alpha^\beta G(t,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

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Now

$$\begin{aligned} Mu(t) =& t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds \right.\\ &\left. - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds \right). \end{aligned}$$

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Let

$$v(t) = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds.$$

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$$v'(1) = (\alpha-1)\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds - (\alpha-1)\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} p(s)u(s)ds < 0.$$

So $v(0) = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds > 0$ and

$$\begin{aligned} v'(1) =& (\alpha - 1) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds \\ & - (\alpha - 1) \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} p(s)u(s)ds < 0. \end{aligned}$$

So $M : \mathcal{P} \backslash \{0\} \to \Omega \subset \mathcal{P}^\circ$.

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$$u(0) = u(1) = 0.$$

So the eigenvalues of (1),(3) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (2),(3) are reciprocals of eigenvalues of N, and conversely.

Theorem

Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.

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Theorem

Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(t) \leq q(t)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in the previous theorem associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(t) = q(t)$ on $[0,1]$.

The following theorem is an immediate consequence of the relationship between the eigenvalues of M and (1),(3), and the eigenvalues of N and (2),(3), and the previous two theorems.

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Theorem

Assume the hypotheses of the previous theorem. Then there exists smallest positive eigenvalues λ_1 and λ_2 of (1),(3) and (2),(3), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(t) = q(t)$ for all $t \in [0, 1]$.