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An Upper bound on the Spectral p-norms of Tensors and Matrix Permanent

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An Upper bound on the Spectral p-norms of Tensors and Matrix Permanent

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Abstract

In this paper, we study the combinatorial properties of spectral p-norms for hypermatrices. We obtain an upper bound for these norms when the hypermatrices satisfy certain conditions.

For a special, hypercubical tensor, $\mathcal{A}$, where $\mathcal{A}_{i_1,\ldots,i_m} = 0$ if any $i_j = i_k$ for $j \neq k$, and $\mathcal{A}_{i_1,\ldots,i_m} \geq 0$ along with its p-spectral norm, $||\mathcal{A}||_p$, $p \geq 1$, has a developed upper bound. That is, $||\mathcal{A}||_p \leq |\mathcal{A}|_1$. Although this inequality is a working definition for arbitrary tensors, the boundedness of this specified p-norm can be improved upon.

Keywords: Permanent, p-spectral norm, Lagrange Multiplier Method, Hypercubical, Non-negative Tensor, Hadamard Product

1 Introduction

The structure of this paper is as follows. In the first part we have revisited the basic definitions and fundamental results that reflects standard techniques related to the topic. In the next section we present our main theorem and provide examples to indicate its efficiency. In the last section we have incorporated the Hadamard product and related additional results for spectral $p-$norms.

Definition 1.1. A **Hypermatrix** (Tensor) $\mathcal{A} = (A_{i_1,\ldots,i_m})$ is a multi-array of entries $\mathcal{A}_{i_1,\ldots,i_m} \in \mathbb{F}$, where $(i_1 = 1,\ldots,n_1,\ldots,i_m = 1,\ldots,n_m)$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Here, m is called the order of $\mathcal{A}$ and $(n_1,\ldots,n_m)$ is its dimension. If $n_1 = \ldots = n_m = n$, then $\mathcal{A}$ is an mth order, dimension n hypermatrix. The collection of all said hypermatrices is denoted as $\mathcal{T}_{m,n}$.We'll assume that both $m, n \geq 2$.

Let $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{F}^n$. An ℓ_p**-norm**, with $p \geq 1$, is defined as follows:

$$|\mathbf{x}|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \tag{1}$$

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Let $\mathcal{A} \in \mathcal{T}_{m,n}$ and $p \geq 1$ be a real number. The **Spectral p-norm** of $\mathcal{A}$, is defined as

$$||\mathcal{A}||_p = \max\{|L_{\mathcal{A}}(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(n)})| : \ \mathbf{x}^{(i)} \in \mathbb{F}^n, |\mathbf{x}^{(i)}|_p = 1, i = 1, ..., n\} \tag{2}$$

$$L_{\mathcal{A}}(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(n)}) = \sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} x_{i_1}^{\overline{(1)}} ... x_{i_m}^{\overline{(m)}}. \text{ for } \mathbf{x}^{(i)} = (x_1^{(i)}, ..., x_n^{(i)})$$

If $|\mathbf{x}^{(i)}|_p = 1$ for $i = 1, ..., n$ and $||\mathcal{A}||_p = |L_{\mathcal{A}}(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)})|$, then the m-tuple $(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)})$ is referred to as an eigenkit to $||\mathcal{A}||_p$. Given $\mathcal{A} \in \mathcal{T}_{m,n}$, p-norm of a tensor $\mathcal{A}$, with $p \geq 1$, is

$$|\mathcal{A}|_p = \left(\sum_{i_1,...,i_m} |\mathcal{A}_{i_1,...,i_m}|^p \right)^{1/p} \tag{3}$$

Definition 1.2. The **Permanent** of a square matrix, X, is given by $Perm(X) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i\sigma(i)}$, where $X = [x_{ij}]$ of the specified size $n \times n$ and S_n is the group of all permutations σ for n elements.

The following are known and can be found in ([5]). Nevertheless, since they include standard techniques in the study of spectral p-norms, we include their proofs as well.

Proposition 1. $||\mathcal{A}||_p \leq |\mathcal{A}|_1$

Proof. Let $(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)})$ be the eigenkit of the p-norm on $\mathcal{A}$, where $|\mathbf{x}^{(i)}|_p = 1$

$$|\mathbf{x}^{(i)}|_p = 1 \implies \sum_{j=1}^{n} |x_j^{(i)}|^p = 1 \implies |x_j^{(i)}| \leq 1, j = 1, ..., n$$

Then, $\prod_{k=1}^{m} |x_{i_k}^{(k)}| \leq 1$. Also, by the Triangle Inequality,

$$||\mathcal{A}||_p = |L_{\mathcal{A}}(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(n)})| \leq \sum_{i_1,...,i_m} |\mathcal{A}_{i_1,...,i_m}| \left(\prod_{k=1}^{m} |x_{i_k}^{(k)}| \right) \leq \sum_{i_1,...,i_m} |\mathcal{A}_{i_1,...,i_m}| = |\mathcal{A}|_1$$

Hence, $||\mathcal{A}||_p \leq |\mathcal{A}|_1$

□

Proposition 2. $||\mathcal{A}||_1 = |\mathcal{A}|_{max}$

Proof. Recall that $||\mathcal{A}||_1 = \max\{|L_{\mathcal{A}}(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(n)})| : \ |x^{(i)}|_1 = 1\}$. Without loss of generality, let $|\mathcal{A}_{s_1,...,s_n}| = |\mathcal{A}|_{max}$. Then, for eigenkit $(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)})$,

$$||\mathcal{A}||_1 = |L_{\mathcal{A}}(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(n)})| \leq \sum_{i_1,...,i_m} |\mathcal{A}_{i_1,...,i_m}| \left(\prod_{k=1}^{m} |x_{i_k}^{(k)}| \right) \leq |\mathcal{A}_{s_1,...,s_n}| \sum_{i_1,...,i_m} |x_{i_1}^{(1)}||x_{i_2}^{(2)}|...|x_{i_n}^{(n)}|$$

$$\leq |\mathcal{A}_{s_1,...,s_n}|$$

Consequently, $||\mathcal{A}||_1 \leq |\mathcal{A}|_{max}$. Proceeding in the other direction, Let $\mathbf{x}^{(i)}$ be vectors where $x_j^{(i)} = 1$ if $j = s_i$ and $x_j^{(i)} = 0$ otherwise. Then,

$L_{\mathcal{A}}(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(n)}) = \sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} x_{i_1}^{(1)} ... x_{i_m}^{(m)} = \mathcal{A}_{s_1,...,s_m}$. As a result,
$|L_{\mathcal{A}}(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(n)})| = |\mathcal{A}_{s_1,...,s_m}| \leq ||\mathcal{A}||_1$. Consequently, this shows that $||\mathcal{A}||_1 = |\mathcal{A}|_{max}$

□

Proposition 3. Let $\mathcal{A} \in \mathcal{T}_{m,n}$ and $p \geq q \geq 1$, then $||\mathcal{A}||_p \geq ||\mathcal{A}||_q$

Proof. Let $p \geq q \geq 1$. Pick an eigenkit $(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)})$ for $||\mathcal{A}||_q$. Then $|\mathbf{x}^{(i)}|_q = 1$ and since $p \geq q$, $|\mathbf{x}^{(i)}|_p \leq |\mathbf{x}^{(i)}|_q = 1$. In particular, $|x_j^{(i)}| \leq 1$ for all $i = 1, ..., m$ and $j = 1, ..., n$. Now, let $\mathbf{y}^{(i)} = \frac{\mathbf{x}^{(i)}}{|\mathbf{x}^{(i)}|_p}$. As a result, $|\mathbf{y}^{(i)}|_p = 1$ and $x_j^{(i)} = |\mathbf{x}^{(i)}|_p y_j^{(i)}$. Then,

$$
\begin{aligned}
||\mathcal{A}||_q &= \left| \sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} x_{i_1}^{(1)} ... x_{i_m}^{(m)} \right| = |\mathbf{x}^{(1)}|_p ... |\mathbf{x}^{(m)}|_p \left| \sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} y_{i_1}^{(1)} ... y_{i_m}^{(m)} \right| \\
&\leq \left| \sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} y_{i_1}^{(1)} ... y_{i_m}^{(m)} \right| \leq ||\mathcal{A}||_p
\end{aligned}
$$

Consequently, $||\mathcal{A}||_p \geq ||\mathcal{A}||_q$.

□

Theorem 1. ([8]) Given a square $n \times n$ matrix $X = [\mathbf{x}^{(1)}\ \mathbf{x}^{(2)}\ ...\ \mathbf{x}^{(n)}]$, the following inequality is satisfied,

$$|Perm(X)| \leq \frac{n!}{n^{n/2}} \prod_{i=1}^{n} |\mathbf{x}^{(i)}|_2$$

where $Perm(X)$ is the Permanent of the matrix X and $|\mathbf{x}^{(i)}|_2$ is the 2-norm of the vector $\mathbf{x}^{(i)}$.

Spectral $p-$ norms of tensors are natural extensions of spectral 2-norms of matrices. For hypermatrices the idea of spectral and nuclear norms are defined explicitly in ([1], [2]). It is worth noting that norm computations for hypermatrices for $p > 2$ is NP-hard ([4]). Due to its close connections to hypergraph theory, spectral $p-$ norms of hypermatrices have been studied extensively (see [5],[6],[7], and reference therein).

2 Methods

Definition 2.1. A hypermatrix $\mathcal{A} = (\mathcal{A}_{i_1,...,i_m}) \in \mathcal{T}_{m,n}$ is non-negative if $\mathcal{A}_{i_1,...,i_m} \geq 0$. We write $\mathcal{A} \geq 0$ if it is a non-negative tensor.

Proposition 4. If $\mathcal{A}$ is a non-negative tensor, $\mathcal{A} \in \mathcal{T}_{m,n}$, then $||\mathcal{A}||_p = |L_{\mathcal{A}}(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)})|$ where $x_j^{(i)} \geq 0$, for $i = 1, ..., m$, $j = 1, ..., n$ and $|x^{(i)}|_p = 1$

Proof. Recall the definition of the p-spectral norm and the triangle inequality. Choose $(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(n)})$ to be the eigenkit for $||\mathcal{A}||_p$

$$||\mathcal{A}||_p = |L_{\mathcal{A}}(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(n)})| = \left|\sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} x_{i_1}^{\overline{(1)}} ... x_{i_m}^{\overline{(m)}}\right| \leq \sum_{i_1,...,i_m} |\mathcal{A}_{i_1,...,i_m}||x_{i_1}^{\overline{(1)}}| ... |x_{i_m}^{\overline{(m)}}|$$

$$= \sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} |x_{i_1}^{\overline{(1)}}| ... |x_{i_m}^{\overline{(m)}}| = \sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} |x_{i_1}^{(1)}| ... |x_{i_m}^{(m)}|$$

For the other direction, a set of vectors $\mathbf{y}^{(i)}$ may be defined such that $y_j^{(i)} = |x_j^{(i)}| \geq 0$,

$$|L_{\mathcal{A}}(\mathbf{y}^{(1)}, ..., \mathbf{y}^{(n)})| = L_{\mathcal{A}}(\mathbf{y}^{(1)}, ..., \mathbf{y}^{(n)}) = \sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} y_{i_1}^{\overline{(1)}} ... y_{i_m}^{\overline{(m)}}$$

$$= \sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} y_{i_1}^{(1)} ... y_{i_m}^{(m)} = \sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} |x_{i_1}^{(1)}| ... |x_{i_m}^{(m)}| = \sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} |x_{i_1}^{\overline{(1)}}| ... |x_{i_m}^{\overline{(m)}}|$$

$$= \sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} |x_{i_1}^{\overline{(1)}} ... x_{i_m}^{\overline{(m)}}| = \sum_{i_1,...,i_m} |\mathcal{A}_{i_1,...,i_m} x_{i_1}^{\overline{(1)}} ... x_{i_m}^{\overline{(m)}}| \geq ||\mathcal{A}||_p \geq \left|\sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} x_{i_1}^{\overline{(1)}} ... x_{i_m}^{\overline{(m)}}\right|$$

Hence, $|L_{\mathcal{A}}(\mathbf{y}^{(1)}, ..., \mathbf{y}^{(m)})| \geq ||\mathcal{A}||_p$. There is only equality iff $x_j^{(i)} \geq 0$ for all indices, that is, all of the components for each $\mathbf{x}^{(i)}$ are non-negative. □

With all these theorems outlined, the goal of this paper can then be properly examined. Although it is known that $||\mathcal{A}||_p \leq |\mathcal{A}|_{max}$, another upper bound can be calculated with specific suppositions.

Theorem 2. Let $\mathcal{A} \in \mathcal{T}_{n,n}$ be a nonnegative tensor such that, $\mathcal{A}_{i_1,...,i_n} = 0$ if any $i_j = i_k$ for $j \neq k$. Then for any $p \geq 2$,

$$||\mathcal{A}||_p \leq |\mathcal{A}|_{max} \frac{n!}{n^{n/p}}$$

Proof. Let $(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(n)})$ be the eigenkit for $||\mathcal{A}||_p$. By **Proposition 4.**, we can have $x_j^{(i)} \geq 0$

$$||\mathcal{A}||_p = \sum_{i_1,...,i_n} \mathcal{A}_{i_1,...,i_n} x_{i_1}^{(1)} ... x_{i_n}^{(n)} \leq |\mathcal{A}|_{max} \sum_{i_1,...,i_n} x_{i_1}^{(1)} ... x_{i_n}^{(n)}.$$

The sum $\sum_{i_1,...,i_n} x_{i_1}^{(1)} ... x_{i_n}^{(n)}$ is the **Permanent** of the matrix below,

$$X = [x_{ij}] = [\mathbf{x}^{(1)}\ \mathbf{x}^{(2)}\ ...\ \mathbf{x}^{(n)}] = \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & ... & x_1^{(n)} \\ x_2^{(1)} & x_2^{(2)} & ... & x_2^{(n)} \\ ... & ... & ... & ... \\ x_n^{(1)} & x_n^{(2)} & ... & x_n^{(n)} \end{bmatrix}$$

of size $n \times n$. By **Theorem 1.**, the Permanent of this matrix satisifes the inequality, $|Perm(X)| \leq \frac{n!}{n^{n/2}} \prod_{i=1}^{n} |\mathbf{x}^{(i)}|_2$. Combining these two distinct inequalities together yields

$$||A||_p \leq |\mathcal{A}|_{max} \sum_{i_1,...,i_n} x_{i_1}^{(1)}...x_{i_n}^{(n)} = |\mathcal{A}|_{max} \frac{n!}{n^{n/2}} \prod_{i=1}^{n} |\mathbf{x}^{(i)}|_2 \tag{4}$$

Now, we use the **Lagrange Multiplier Method** to obtain a maximum value of $\prod_{i=1}^{n} |\mathbf{x}^{(i)}|_2$.

$|\mathbf{x}^{(i)}|_2$ may be optimized to the given constraint that $|\mathbf{x}^{(i)}|_p = 1$. Algebraically, this is represented as $|\mathbf{x}^{(i)}|_2^2 - \lambda(|\mathbf{x}^{(i)}|_p^p - 1) = \sum_{j=1}^{n}(x_j^{(i)})^2 - \lambda\left(\sum_{j=1}^{n}(x_j^{(i)})^p - 1\right) = 0$. Decoupling the system and deriving with respect to $x_j^{(i)}$ and λ yields the following:

$$2x_j^{(i)} - \lambda p(x_j^{(i)})^{p-1} = 0 \tag{5}$$

$$-(x_j^{(i)})^p + 1 = 0 \tag{6}$$

(6) reveals that $(x_j^{(i)})^p = 1$, for all indices i, j. Further investigation in (5) shows

$$2x_j^{(i)} - \lambda p(x_j^{(i)})^{p-1} = 0 \implies 2(x_j^{(i)})^2 - \lambda p(x_j^{(i)})^p = 0 \implies 2(x_j^{(i)})^2 = \lambda p$$

Equivalently, $|x_j^{(i)}| = x_j^{(i)} = \left(\frac{\lambda p}{2}\right)^{1/2}$. This implies that $(x_j^{(i)})^p = \left(\frac{\lambda p}{2}\right)^{p/2}$ $\implies \sum_{j=1}^{n}(x_j^{(i)})^p = n\left(\frac{\lambda p}{2}\right)^{p/2} = 1$ due to the p-norm constraint $|\mathbf{x}^{(i)}|_p^p = 1$. Solving for λ shows that

$$\lambda = n^{-2/p}\left(\frac{2}{p}\right) = (x_j^{(i)})^2\left(\frac{2}{p}\right) \implies (x_j^{(i)})^2 = n^{-2/p}$$

With this result, $|x^{(i)}|_2$ can be calculated for each i.

$$(x_j^{(i)})^2 = n^{-2/p} \implies \sum_{j=1}^{n}(x_j^{(i)})^2 = n^{1-2/p} \implies |\mathbf{x}^{(i)}|_2 = n^{1/2-1/p}$$

Relating back to the earlier inequality in (4)

$$||\mathcal{A}||_p \leq |\mathcal{A}|_{max} \frac{n!}{n^{n/2}} \prod_{i=1}^{n} |\mathbf{x}^{(i)}|_2 = |\mathcal{A}|_{max} \frac{n!}{n^{n/2}} n^{n/2-n/p} = |\mathcal{A}|_{max} \frac{n!}{n^{n/p}}$$

This completes the proof. □

We can relax some requirements in the previous theorem. Let $\mathcal{A} \in \mathcal{T}_{m,n}$ be any tensor. Define $|\mathcal{A}|$ as $|\mathcal{A}|_{i_1,\dots,i_m} = |\mathcal{A}_{i_1,\dots,i_m}|$ for all $i_1, \dots, i_m$. Then, $|\mathcal{A}| \geq 0$.

Corollary 2.1. Let $\mathcal{A} \in \mathcal{T}_{n,n}$ which satisfies the requirements of **Theorem 2.** except $\mathcal{A}$ is not necessarily non-negative and it may have complex entries. Then,

$$|| \, |\mathcal{A}| \, ||_p \leq |\mathcal{A}|_{max} \frac{n!}{n^{n/p}}$$

for all $p \geq 2$

Proof. Note that $|\mathcal{A}| \geq 0$ and that $\left||\mathcal{A}|\right|_{max} = \max_{i_1,\dots,i_n} \left\{\left| \, |\mathcal{A}|_{i_1,\dots,i_n}\right|\right\} = \max_{i_1,\dots,i_n} \{ \, |\mathcal{A}_{i_1,\dots,i_n}|\}$ $= |\mathcal{A}|_{max}$. Consequently, the result follows directly from **Theorem 2.**

□

Now, we can present some examples to demonstrate the efficiency of the upper bound found in **Theorem 2.**

Example 1. Let $\mathcal{A} \in \mathcal{T}_{n,n}$ with $\mathcal{A}_{i_1,\dots,i_n} = 1$ if all $i_1, \dots, i_n$ are distinct and $\mathcal{A}_{i_1,\dots,i_n} = 0$ otherwise. Then $|\mathcal{A}|_{max} = 1$. For any $p \geq 2$, we have $||\mathcal{A}||_p \leq |\mathcal{A}|_{max} \frac{n!}{n^{n/p}} = \frac{n!}{n^{n/p}}$. Considering that $|\mathcal{A}|_1 = \sum_{i_1,\dots,i_n} \mathcal{A}_{i_1,\dots,i_n} = n!$, then we clearly have $||\mathcal{A}||_p \leq \frac{n!}{n^{n/p}} < n!$

For the following example, we define the row sum of a tensor. If $\mathcal{A} \in \mathcal{T}_{n,n}$, then the ith row sum of $\mathcal{A}$, denoted by r_i, is $r_i = \sum_{i_2,\dots,i_n} \mathcal{A}_{i,i_2,\dots,i_n}$.

Example 2. Let $\mathcal{A} \in \mathcal{T}_{6,6}$ and $\mathcal{A} \geq 0$ such that $\mathcal{A}$ satisfies the conditions of **Theorem 2.** and $r_i = r > 0$ for $i = 1, 2, \dots, 6$. Then, for $p = 2$,

$$||\mathcal{A}||_2 \leq |\mathcal{A}|_{max} \frac{6!}{6^{6/2}} = |\mathcal{A}|_{max} \frac{6!}{6^3} \leq \frac{10}{3} r$$

and $|\mathcal{A}|_1 = 6r$. Clearly, $||\mathcal{A}||_2 \leq \frac{10}{3} r < 6r$

3 Additional Results on Spectral p-norms

Let $\mathcal{A}, \mathcal{B} \in \mathcal{T}_{m,n}$. The **Hadamard product** of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \circ \mathcal{B}$, is the tensor $(\mathcal{A} \circ \mathcal{B})_{i_1,\dots,i_m} = \mathcal{A}_{i_1,\dots,i_m} \mathcal{B}_{i_1,\dots,i_m}$

Proposition 5. Let $\mathcal{A}, \mathcal{B} \in \mathcal{T}_{m,n}$. For any $p \geq 1$,

$$||\mathcal{A} \circ \mathcal{B}||_p \leq |\mathcal{A}|_2 |\mathcal{B}|_2$$

Proof. Let $(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)})$ be eigenkit for $||\mathcal{A} \circ \mathcal{B}||_p$. Then,
$$||\mathcal{A} \circ \mathcal{B}||_p = \left| \sum_{i_1,...,i_m} \mathcal{A}_{i_1,...,i_m} \mathcal{B}_{i_1,...,i_m} \overline{x_{i_1}^{(1)}} ... \overline{x_{i_m}^{(m)}} \right| \leq \sum_{i_1,...,i_m} |\mathcal{A}_{i_1,...,i_m}||\mathcal{B}_{i_1,...,i_m}||\overline{x_{i_1}^{(1)}}|...|\overline{x_{i_m}^{(m)}}|$$

Since $|x_{i_k}^{(k)}| \geq 0$, we can let $|x_{i_k}^{(k)}| = (y_{i_k}^{(k)})^2 \geq 0$, where $y_{i_k}^{(k)}$ are non-negative real numbers. Replacing $|x_{i_k}^{(k)}|$ we find the following,

$$\begin{aligned} ||\mathcal{A} \circ \mathcal{B}||_p &\leq \sum_{i_1,...,i_m} |\mathcal{A}_{i_1,...,i_m}||\mathcal{B}_{i_1,...,i_m}|(y_{i_1}^{(1)})^2 ... (y_{i_k}^{(k)})^2 \\ &\leq \sum_{i_1,...,i_m} \left(|\mathcal{A}_{i_1,...,i_m}||y_{i_1}^{(1)}|...|y_{i_k}^{(k)}| \right) \left(|\mathcal{B}_{i_1,...,i_m}||y_{i_1}^{(1)}|...|y_{i_k}^{(k)}| \right) \\ &\leq \sum_{i_1,...,i_m} \left(|\mathcal{A}_{i_1,...,i_m}|^2 \prod_{k=1}^{m} |y_{i_k}^{(k)}|^2 \right)^{1/2} \left(|\mathcal{B}_{i_1,...,i_m}|^2 \prod_{k=1}^{m} |y_{i_k}^{(k)}|^2 \right)^{1/2} \end{aligned}$$

Let $\alpha = \prod_{k=1}^{m} |y_{i_k}^{(k)}|^2 \leq 1$. Then, we find
$$||\mathcal{A} \circ \mathcal{B}||_p \leq \sum_{i_1,...,i_m} \left(|\mathcal{A}_{i_1,...,i_m}|^2 \alpha \right)^{1/2} \left(|\mathcal{B}_{i_1,...,i_m}|^2 \alpha \right)^{1/2}$$

Recall that for any $r, s > 0$, we have
$$\left(\sum_{k=1}^{n} |x_k|^r |y_k|^s \right)^{r+s} \leq \left(\sum_{k=1}^{n} |x_k|^{r+s} \right)^r \left(\sum_{k=1}^{n} |y_k|^{r+s} \right)^s$$
Now, using this last inequality by choosing $r = s = 1/2$,
$$\begin{aligned} ||\mathcal{A} \circ \mathcal{B}||_p &\leq \left(\sum_{i_1,...,i_m} |\mathcal{A}_{i_1,...,i_m}|^2 \alpha \right)^{1/2} \left(\sum_{i_1,...,i_m} |\mathcal{B}_{i_1,...,i_m}|^2 \alpha \right)^{1/2} \\ &\leq \left(\sum_{i_1,...,i_m} |\mathcal{A}_{i_1,...,i_m}|^2 \right)^{1/2} \left(\sum_{i_1,...,i_m} |\mathcal{B}_{i_1,...,i_m}|^2 \right)^{1/2} \end{aligned}$$

Consequently, $||\mathcal{A} \circ \mathcal{B}||_p \leq |\mathcal{A}|_2 |\mathcal{B}|_2$

□

Corollary 2.2. Let $\mathcal{A} \in \mathcal{T}_{m,n}$. Then, $||\mathcal{A} \circ \mathcal{A}||_p \leq |\mathcal{A}|_2^2$

Proof. By replacing $B = A$ using **Proposition 5.**, we see that the result is achieved.

□

Proposition 6. Let $\mathcal{A} \in \mathcal{T}_{m,n}$ where $\mathcal{A}_{i_1,...,i_m} \in \mathbb{R}$. Then for any $p \geq 1$, $||\mathcal{A} \circ \mathcal{A}||_p \leq \left\| |\mathcal{A}| \right\|_{2p}^2$, even if $\mathcal{A}$ is not non-negative.

Proof. Let $(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)})$ be eigenkit for $||\mathcal{A} \circ \mathcal{A}||_p$. Since the Hadamard product is an entry-wise multiplication, $\mathcal{A} \circ \mathcal{A} \geq 0$, even if $\mathcal{A}$ is not necessarily non-negative. Then, by **Proposition 4.**, we can assume $x_j^{(i)} \geq 0$. We observe the following,

$||\mathcal{A} \circ \mathcal{A}||_p = \sum_{i_1,...,i_m} (\mathcal{A}_{i_1,...,i_m})^2 x_{i_1}^{(1)} ... x_{i_m}^{(m)}$. Now, we can choose $y_j^{(i)}$ such that $x_j^{(i)} = (y_j^{(i)})^2$.

Then,

$$||\mathcal{A} \circ \mathcal{A}||_p = \sum_{i_1,...,i_m} (\mathcal{A}_{i_1,...,i_m})^2 (y_{i_1}^{(1)})^2 ... (y_{i_m}^{(m)})^2 = \sum_{i_1,...,i_m} \left(|\mathcal{A}_{i_1,...,i_m}| (y_{i_1}^{(1)}) ... (y_{i_m}^{(m)})\right)^2$$

$$\leq \left(\sum_{i_1,...,i_m} |\mathcal{A}_{i_1,...,i_m}| y_{i_1}^{(1)} ... y_{i_m}^{(m)} \right)^2$$

Note that for any i, $\sum_{j=1}^{n} |y_j^{(i)}|^{2p} = \sum_{j=1}^{n} |x_j^{(i)}|^p = |\mathbf{x}^{(i)}|_p^p = 1$. This implies that $|\mathbf{y}^{(i)}|_{2p} = 1$.

Then,

$$||\mathcal{A} \circ \mathcal{A}||_p \leq \left(\sum_{i_1,...,i_m} |\mathcal{A}_{i_1,...,i_m}| y_{i_1}^{(1)} ... y_{i_m}^{(m)} \right)^2 \leq \left\| |\mathcal{A}| \right\|_{2p}^2$$

□

Corollary 2.3. Let $\mathcal{A} \in \mathcal{T}_{m,n}$ with $A \geq 0$. Then $||\mathcal{A} \circ \mathcal{A}||_p \leq ||\mathcal{A}||_{2p}^2$, for all $p \geq 1$.

Proof. Since $\mathcal{A} \geq 0$. This implies that $|\mathcal{A}| = \mathcal{A}$. Thus using **Proposition 6.** yields the result. □

Corollary 2.4. Let $\mathcal{A} \in \mathcal{T}_{n,n}$ and that $\mathcal{A}$ satisfies the conditions in **Theorem 2.**. Then for $p \geq 1$

$$||\mathcal{A} \circ \mathcal{A}||_p \leq \left(|\mathcal{A}|_{max} \frac{n!}{n^{n/2p}} \right)^2$$

Proof. Using the previous corollary and **Theorem 1.** completes the proof.

□

4 Conclusion

The theorems, corollaries, and propositions helped achieve the goal of this paper, which was to improve upon the boundedness of the p-norm of hypercubical tensors. With further investigation and proofs, it can be shown that the p-norm of any tensor $\mathcal{B} \in \mathcal{T}_{m,n}$, $||\mathcal{B}||_p$, can be determined by the existence of a unique, nonnegative vector $\mathbf{x}$ and the eigenkit equation below

$$\left(||\mathcal{B}||_p \mathcal{I} - \mathcal{A} \right) \mathbf{x}^{[n-1]} = ||\mathcal{B}||_p$$

where $\mathcal{I}$ is the Identity Tensor defined previously and $\mathcal{A} \geq 0$. To approach this problem further, types of eigenvalues and related concepts would have to be explored. A number $\lambda \in \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is referred to as an **eigenvalue** of the tensor $\mathcal{A}$ if for any nonzero vector **x** and itself, the following is true

$$(\mathcal{A}\mathbf{x}^{m-1})_i = \lambda x_i^{m-1}, \ \forall i = 1, \ldots, n$$

With this in mind, the set of all of eigenvalues of $\mathcal{A}$ is referred to as the **spectrum** of $\mathcal{A}$. Consequently, the **spectral radius** of the $\mathcal{A}$ is defined as the largest modulus of the eigenvalues of $\mathcal{A}$, or simply $\rho(\mathcal{A}) = \max\{|\lambda|\}$.

The eigenkit equation above would imply that $\rho(\mathcal{A}) = 0$. In fact, Some of the algorithms outlined in (see [3]) are useful for solving this and similar problems. The CDN, NQZ, and LZI methods are just some of the useful algorithms that would be helpful in this discussion. If it weren't for time constraints, then these would've been further explored. Nonetheless, these procedures would be a topic of interest for a second paper as it remains pertinent to both this paper and other computational system of polynomial equations involved in Ordinary Differential Equations.

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