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When Two Wrongs Made A Right: A Classroom Scenario of Critial Thinking as Problem Solving

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WHEN TWO WRONGS MADE A RIGHT: A Classroom Scenario of Critical Thinking as Problem Solving

> Of fate two wrongs infer one right... (C. Ackers for J. Wilford, ed. 1734. "The London Magazine, or, Gentleman's Monthly Intelligencer, Volume 3")

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Critical thinking in Mathematical problem solving

Educators from kindergarten through college often stress the importance of teaching critical thinking within all academic content areas (Foundation for Critical Thinking, 2007, 2013). As indicated by the position statements of the National Council of Teachers of Mathematics, high quality mathematics education before the first grade should use curriculum and teaching practices that strengthen children's problem-solving and reasoning processes as well as representing, communicating, and connecting mathematical ideas" The joint position statement of NAEYC and the National Council of" (NAEYC & NCTM [2002] 2010, 3). Through the educational and academic institutions critical thinking is identified as an important outcome for achieving the higher orders of learning upon successful completion of a course, a promotion, or a degree (Humphreys, 2013; Jenkins & Cutchens, 2011). Although there are numerous definitions of critical thinking, the authors have selected the definition by Scriven & Paul, 2008 as "the intellectually disciplined process of actively and skillfully conceptualizing, applying, synthesizing, and/or evaluating information gathered from, or generated by, observation, experience, reflection reasoning, or communication as a guide to belief and action" (Scriven & Paul, 2008). Instructors should teach problem solving within the context of mathematics instruction and engage students in critical thinking by thoughtful questions with discussion of alternative results. Teaching preschool children to problem solve and engage in critical thinking in the context of mathematics instruction requires a series of thoughtful and informed decisions.

MAKING THE CASE IN THE CLASSROOM: An actual historical classroom example

The student made two mistakes, forgetting to extract a square root, adapting to innovation, and flexibility as students usually do. Nevertheless, he immediately realized how to fix them. The result was a simple way to construct Pythagorean triples with an insightful geometric mnemonic rule. We present the following story for accuracy of its mathematical content.

Ms. Angel asked her students to find at least one more Pythagorean triple besides 3, 4, and 5. Nobody came up with one, so Ms. Angel asked Mike to come to the board. She knew Mike was a skilled student, and with her guidance he would be able to construct one such triple.

"Let's try to find a Pythagorean triple, a right triangle with integer sides," Ms. Angel said.

Mike was completely confused. He could not remember mathematical concepts well, but utilizing his critical thinking skills, he was not too embarrassed to ask questions: "Why did you call them Pythagorean?" Mike asked.

"They obey the Pythagorean Theorem," Ms. Angel responded.

Mike continued asking, "What was the Pythagorean Theorem about?"

"The square of the hypotenuse equals the sum of the squares of the legs."

Mike was not satisfied with the explanation, so he remarked, "This is complicated."

"Well, we have the simplest example of a Pythagorean triple: 3, 4, 5," said Ms. Angel.

"That's really nice. What about 1, 2, 3? May I draw it?"

"You may try, but it's not going to be a Pythagorean triple."

"Why not?"

"Because there is no such right triangle with sides 1, 2, 3. It's not even a triangle, but three overlapping segments."

"What about 4, 5, 6?"

"No again, it's not a right triangle."

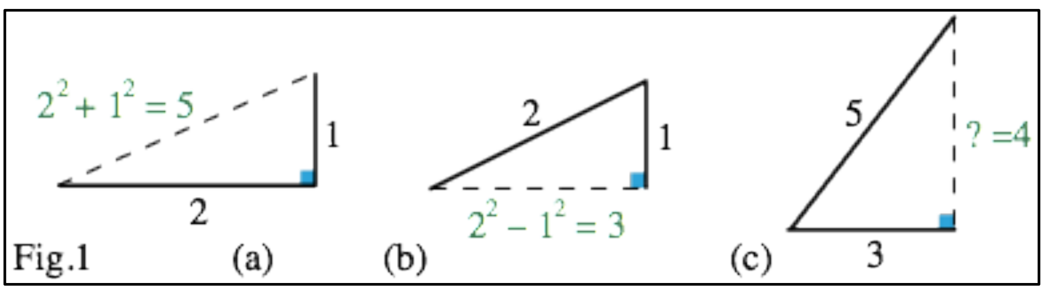
"How should I know that?"

"Because 4 squared plus 5 squared does not equal 6 squared."

"I know how to square numbers, I just draw a square, with that number of unit squares on the sides, and count the number of unit squares inside it." Mike eagerly showed what he knew.

"That's very good, Mike!" Ms. Angel commented. "Back to Pythagorean triples. Do you know another example besides the 3, 4, and 5?"

"Um, let me try. I draw two legs, say, 1 and 2. I square them, 1 and 4, and then add, making 5. That's the hypotenuse of the Pythagorean 3, 4, 5 triangle."



"Not really. You forgot to take the square root; the hypotenuse is a square root of 5 which is not an integer. So, try other possible legs."

Embarrassed, Mike replied, "Sorry, I forgot about the square root."

By now, Mike got really curious and asked, "What if I take the difference of the squares that we already have? The difference of 4 and 1 is 3. That's the leg!" Mike exclaimed.

"Again, you forgot to take the square root!"

"But look, earlier I found the hypotenuse, 5. And now I found the leg 3 of the same 3, 4, 5 Pythagorean triple."

"What about the other leg?" Ms. Angel pressed on.

"I don't know...isn't it automatically determined?"

"Yes it is. But it may not give you a Pythagorean triple ... hmm ... but wait a minute. Obviously, it is 4," Ms. Angel replied.

"Are you sure? That's really cool!"

"That's very interesting! You started with two wrong legs, 1 and 2, and obtained two new correct legs, 3 and 4," smiled Ms. Angel.

"But I had 5 before, so I created the 3, 4, 5 Pythagorean triple by starting with 1 and 2."

"Yes, but this was just a coincidence. Do you want to try another example?"

Mike got really interested and said, "Okay, give me two numbers for the legs."

"Take 3 and 4."

"But, we just did it."

"No, you started with 1 and 2. Now I want you to start with 3 and 4 and go on your way. You will definitely end up with the hypotenuse not being 5, because 5 is the hypotenuse of the 3, 4, 5 triangle. Your hypotenuse is 25, which is 3 squared plus 4 squared."

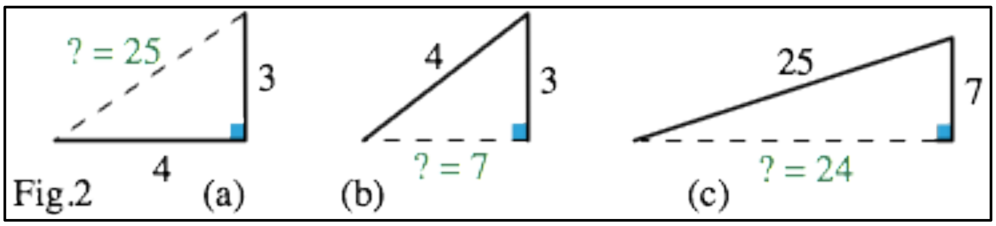


Fig.2

"And that will be wrong?"

"You got it!"

"I am sure you are right, but let me understand…. 25 was the sum of 3 and 4 squared."

"That's correct, Mike!"

"Well, 3 squared is 9, and 4 squared is 16; the sum is 25. But, now I will take their difference which is 16 - 9 = 7. The leg is 7, and not 3 or 4."

"What do you mean, Mike? Aren't you constructing the 3, 4, 5 triangle?"

"No, I am not."

"Then what is the second leg, Mike?"

"We should apply the Pythagorean Theorem."

"Let's try it: $25^2 - 7^2 = 625 - 49 = 576$," Ms. Angel helped to calculate.

"That's too large for a leg," announced Mike.

"We didn't finish yet, Mike. Take the square root ... WOW!"

"Is there something wrong?"

"You are fortunate; it's 24, a correct value. But something is very wrong here."

Beginning to feel comfortable, Mike suggested to Ms. Angel, "Let's try another example, say, 2 and 5. Square them: 2 squared is 4, and 5 squared is 25."

"The sum is 29, that's the hypotenuse," Mike continued. "The difference is 21; that's a leg. The other leg is determined. In fact, I know how to find it. I just double the product of my initial legs, twice 2 times 5 gives $2\times2\times5=20$," Mike wrote on the board. "That's the second leg!"

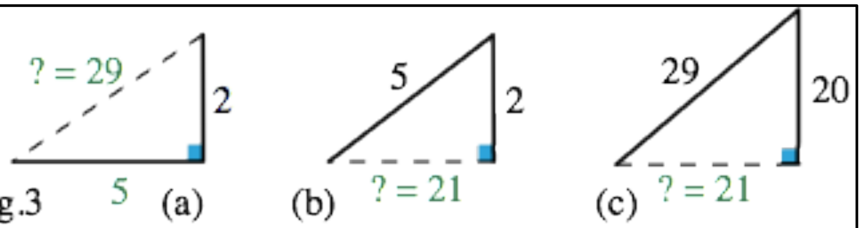


Fig.3

"How did you come up with the double legs product, Mike?"

"Well, I need the difference of two squares. I factored it in my head into their sum and difference. I knew that one of them is a sum and the other is a difference; I could visualize the terms canceling each other and obtaining the product of doubled squares. Because each term is a square of the starting numbers, I saw the product as the double product squared. I don't know how to extract square roots, but I know that the square root of a squared whole number is exactly that number itself."

"That sounds reasonable. Let me check: $20^2 + 21^2$ is 841; that is, 29^2. That's correct, Mike."

"Ms. Angel, I think I did a good job with my homework, right?"

"No, you didn't. If you did, you would not have made this cute discovery."

"What discovery? Isn't this known?"

"The result is known, but not your approach! You made a mistake, or guess, and then fixed it. And you did that not just once, but twice!"

"How many Pythagorean triples are there, Ms. Angel?"

"That number is doubly infinite."

"Can I get all of them?"

"Using your method? Yes, you can!"

"That sounds great!"

"The question is how to prove that your method always works, Mike!"

"We just check the result." Mike replied after a short pause.

"Checking is not a proof in mathematics, although…"

"I mean, checking the general case, for any two starting whole numbers for legs."

"You just gave an idea for a proof. Take two arbitrary integers, call them m and n. The hypotenuse is m squared plus n squared. Take m > n. Then one of the legs is m-squared minus n-squared. Now, if you square these two values and subtract, you will get the

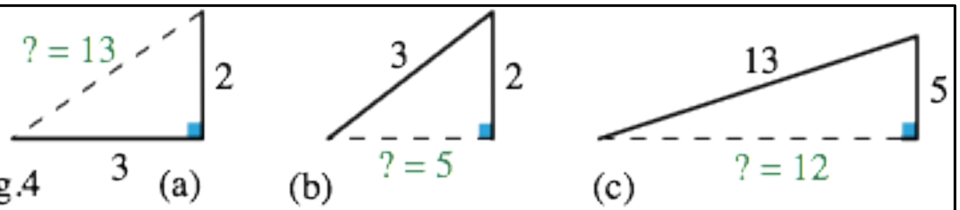


Fig.4 (a) (b) (c)

square of the doubled product, 2mn. This proves the sufficiency of your method." Ms. Angel continued, "I don't know how to prove the necessity, but, I know that this is a known Euclid's two-parametric representation for Pythagorean triples."

"I was sure this is known."

"But your method is also very educational and fun! Isn't it?" She asked the class.

"Yes, it is!" The entire class shouted. "What we like the most is the clearer visual geometric arguments," several students commented. "It helps to remember what to do."

"Then, try this at home. Last time no one came up with any new Pythagorean triple; but now I am asking you to create a dozen of your own Pythagorean triples. It will take less than an hour; it's such an easy homework assignment!" replied Ms. Angel. "I will check the dozen of the Pythagorean triples known to the ancient Babylonians. Some of them involve huge numbers. I was always curious how they obtained these. With guessing and checking it seems impossible, but with Mike's guessing and fixing it's very possible."

"We can construct really large Pythagorean triples," commented some students.

Ms. Angel continued, "I am very pleased with today's work. I also learned some other things that I never knew before: *In Mike-Euclid's Pythagorean triples, the hypotenuse itself is a sum of two square integers*; besides the fact that its square is a sum of squares of two integers."

Then she added. “I didn’t realize before that also in *Mike-Euclid’s Pythagorean triples, one of the legs itself is a difference of two square integers*; besides the fact that its square is a difference of squares of two integers.”

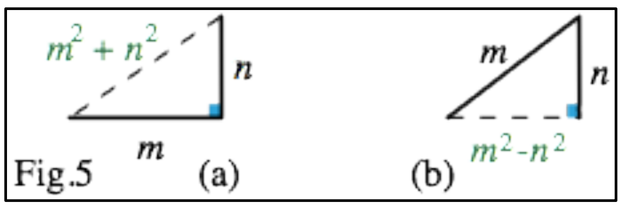
“The nice thing is that no square root need be extracted.” Mike commented proudly.

Ms. Angel decided to summarize the discovery; “We still have ten minutes before the end of class. Let me sketch on the board the summary of the things we learned today. And please make your notes in the notebooks.”

Summary: Pairs of wrongs make all right Pythagorean triples.

1. Start with any two integers, and square them.

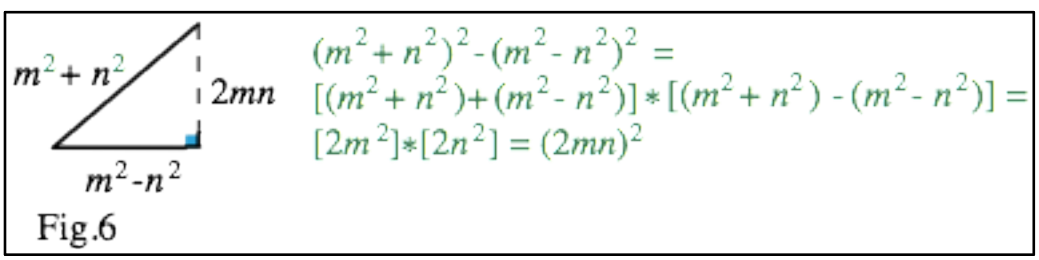
2. Add the squares to get the hypotenuse, but forget to take the square root (Fig. 5a).



3. Subtract the two squares to get a leg, but forget to take the square root (Fig. 5b).

4. The other leg is determined automatically. It is the double product of the starting numbers.

5. For a proof, check the algebra with general notations, using factoring (Fig. 6).



6. A pseudo-geometric interpretation makes it easy for us to remember the Euclid’s algorithm.

7. The three sides in Fig. 6, together with their multiples, generate all the Pythagorean triples.

Concluding remarks:

Sixty years later Mike extended his insightful geometric vision with doing wrongs and taking no roots to higher dimensional spaces, thus generating correct Pythagorean n-tuples, quadruples, pentuples, sextuples, etc. Their known standard formulas are too complicated to memorize. However, the critical thinking technique used by the teacher provides the heuristic teaching approach, which trains the student's mind to become an independent thinker.

This scenario included examples of all the critical thinking factors discussed and needed for embracing new thinking for problem-solving, which is a basis for self-empowerment and enhancement of leadership in all nations. Mathematics teachers can change lives by assisting students to become critical thinkers/problem solvers who will be ready to assume the roles of future leaders, change, and innovation in our expanding global networked society. At an increasing speed, globalization is changing work settings and nonwork environments, and it demands new leaders to make decisions and solve problems often and quickly. Chartrand, Ishikawa, and Flander (2009) cited The Conference Board studies, which indicated that 70% of employees with a high school education were deficient in critical thinking skills and even 10% of college graduates lacked critical thinking skills.

There are many activities that demonstrate opportunities to utilize questioning and critical thinking skills within the mathematical courses of instruction. This real-case scenario was written to provide an example on how teachers can incorporate critical thinking into lesson plans, curriculum and classroom activities.

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