Arithmagons

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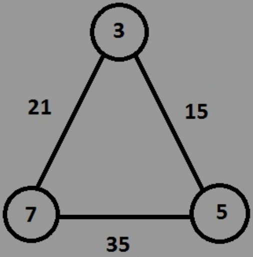
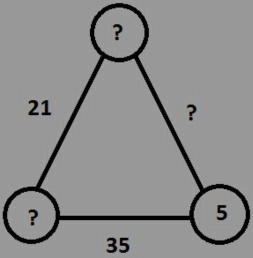
Arithmawhat? What is an arithmagon?

Before giving a formal definition, let us look at an example:



What are these contraptions useful for?

They are useful tools to teach kids basic operations. For example, the arithmagon from before could be presented to a student with some of the entries replaced by unknowns and ask him/her to solve the problem:



A Slightly More Formal Definition

An Arithmagon is a polygonal graph with n sides whose vertices and edges are assigned values $v_1, ..., v_n$ and $e_1, ..., e_n$ respectively, subject to the condition that

$$v_i \star v_{i+1} = e_i \qquad \text{for } 1 \leq i \leq n \tag{1}$$

where $\star$ is an associative and commutative binary operation and $v_{n+1} = v_1$.

We just considered two operations addition and multiplication, though this can be done in more general settings.

Addition or Multiplication?

The first question that we must investigate is whether an arithmagon whose entries in the edges are provided can be solved or not. The original equation (1) becomes

$$v_i v_{i+1} = e_i$$

with multiplication and

$$v_i + v_{i+1} = e_i$$

with addition. The latter system of equation is much more pleasing.

Therefore, we will explore the conditions for solvability for addition. The augmented matrix of the system of equations is of the form:

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & e_1 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 & e_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & e_{n-1} \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 & e_n \end{array}\right)$$

This system has a unique solution whenever n is odd. If n is even, then the system has infinite solutions if

$$\sum_{i=1}^{n/2} v_{2i-1} = \sum_{i=1}^{n/2} v_{2i}$$

What about the multiplicative system

$$v_i v_{i+1} = e_i?$$

By using logarithms, it can be shown that the system has a unique solution whenever n is odd and when n is even it has infinite solutions if

$$\prod_{i=1}^{n/2} \frac{v_{2i-1}}{v_{2i}} = 1.$$

A word must be said about the nature of the solutions. If our arithmagon has vertex entries in $\mathbb{Z}$, do we know that the solutions are in $\mathbb{Z}$?

No.

With addition as the operation we can only guarantee that the solutions will be contained in $\mathbb{Z}/2$.

With multiplication, the solutions would take the form $\sqrt{r}$ for $r \in \mathbb{Q}$.

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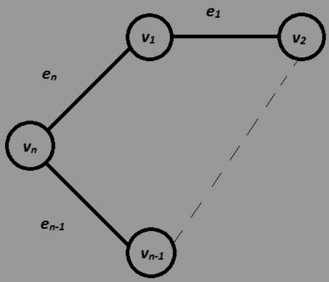
Statement of the Problem

Question

Given two fixed natural numbers $n, N \in \mathbb{N}$, how many n-arithmagons are there such that the product (or sum) of all the values on the edges equals N?

First Observation

Consider the following situation:



Then,

$$N = e_1 \star e_2 \star \cdots \star e_n = (v_1 \star v_2) \star (v_2 \star v_3) \cdots \star (v_n \star v_1)$$
$$= (v_1 \star v_1) \star \cdots \star (v_n \star v_n)$$

Therefore, the number of arithmagons is a perfect square if $\star$ is multiplication or an even number if $\star$ is addition.

From now on we will focus on the multiplicative arithmagons. The result for the additive arithmagons will follow from the multiplicative case.

The first thing that we will notice in this problem is the need for symmetry considerations. We will motivate this with an example.

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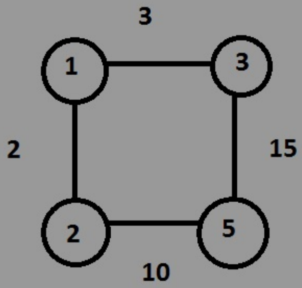
Example with a Square

Suppose that you want to count the number of square arithmagons whose external product is 900. Then we give ourselves a factorization of $\sqrt{900} = 30$, say $30 = 1 \cdot 2 \cdot 3 \cdot 5$ and construct an arithmagon:



Personality test

So, for each factorization of $\sqrt{N}$ we can construct an arithmagon with external product N. Moreover, each arithmagon whose external product is N will yield a factorization of $\sqrt{N}$.

So, if at this point you say "we are done, all we need is to count the factorizations of $\sqrt{N}$ in less than n factors," you are like me... and you get a nice answer in terms of Stirling numbers of the second kind.

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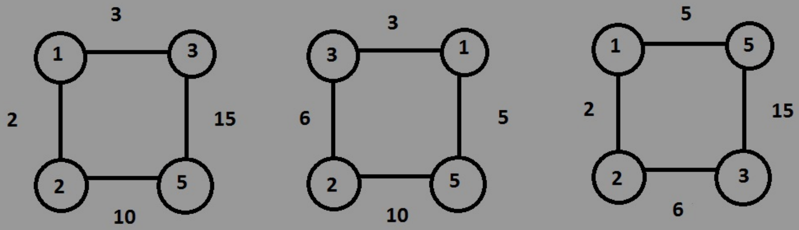
If you are not so sure, you are like Jeff... and you would be correct.

Gladly, Jeff was right and we are not done yet, because you can actually construct three distinct arithmagons with this partition:



Any other multiplicative square arithmagon with vertex entries $1, 2, 3, 5$ will just be a rotation or reflection of these. This is where we get to play with algebra/symmetry.

Quick Algebra Review

We will use the symmetric group on n letters

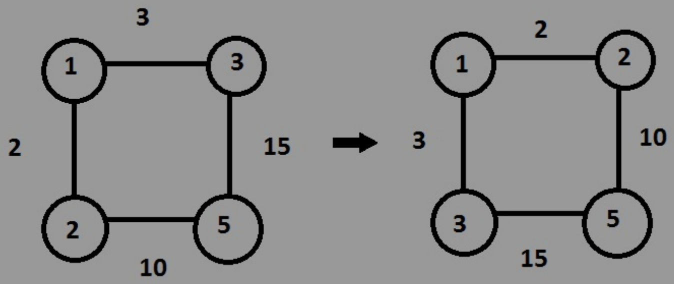
$$S_n := \{\sigma : \{1, ..., n\} \to \{1, ..., n\} \mid \sigma \text{ is bijective}\}$$
$$= \langle (1, n), (1, n-1), ..., (1, 2) \rangle$$

and the group of rotations and reflections of the n-gon, the dihedral group

$$D_{2n} = \langle r := (1, 2, ..., n), s := (1, 2)(3, 4)... \rangle \leq S_n.$$

We notice that each group acts on an arithmagon by permuting the vertices numbered from 1 to n starting at the bottom and left most corner increasing in the counterclockwise direction.

For example, the transposition $(1,3)$ acts on a square arithmagon in the following way:



This transposition gives a new arithmagon from the original because $(1,3) \in S_4 \setminus D_8$.

Turns out that if we have a factorization of $\sqrt{N}$ with n distinct factors (1 can be considered a factor), we can form $|S_n/D_{2n}|$ many distinct arithmagons with this factorization.

To see this, we notice that the action of S_n on $\{1, ..., n\}$ is transitive. Therefore, any possible rearrangement of the entries on the vertices of an arithmagon can be achieved via a permutation in S_n.

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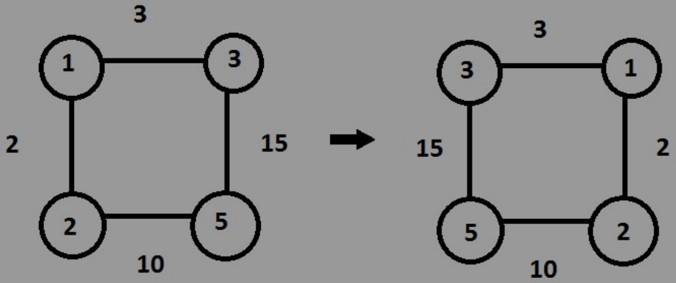
Also notice that any two arithmagons Γ_1 and Γ_2 that are related via an element in $\alpha \in D_{2n}$, that is $\Gamma_2 = \alpha \cdot \Gamma_1$, have the same vertex entries and edge entries, although in different positions.

For example, consider the action of $(1,2)(3,4) \in D_8$ on a square arithmagon,



In other words D_{2n} is the stabilizer of an arithmagon with distinct entries. Therefore, the Orbit-Stabilizer theorem implies that the number of arithmagons corresponding to a factorization of $\sqrt{N}$ with n distinct factors is equal to

$$|S_n/D_{2n}| = \frac{(n-1)!}{2}.$$

So, if we would only be interested in finding the number of arithmagons with n distinct entries whose external product equals N, we would need the number of such factorizations of $\sqrt{N}$ and multiply it by $\frac{(n-1)!}{2}$.

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However, we will not do that just yet. After some symmetry considerations, we will be able to relax this last condition a bit, allowing 1 to appear as many as $n - 1$ times. This will allow for arithmagons with distinct non-identity entries.

To do this we need a rather technical lemma.

Lemma

Let Γ be an n-arithmagon and let G_Γ be the stabilizer of Γ. If l denotes the number of distinct non-identity vertex entries and $k-l$ denotes the number of 1's in the vertices, then the number of arithmagons with the same vertex entries as Γ is given by

$$s(\Gamma) := |S_n \cdot \Gamma| = \frac{n!}{|G_\Gamma|}$$

and

$$|G_\Gamma| = \begin{cases} 2n\prod_{i=l+1}^{k} |[a_i]|! & \text{if } l \geq 3 \\ 2\cdot(n-1)! & \text{if } l = 2 \text{ and } k \leq 3 \\ n! & \text{if } l = 1 \text{ and } k \leq 2 \end{cases}$$

when n is even.

When n is odd, we have

$$|G_\Gamma| = \begin{cases} 2n\prod_{i=l+1}^{k} |[a_i]|! & \text{if } l \geq 2 \\ n! & \text{if } l = 1 \text{ and } k \leq 2 \end{cases}$$

Using this lemma, it is easy to show the following:

Theorem

Let $P_d(l, N)$ denote the number of multiplicative partitions of N in l distinct factors. The number of multiplicative arithmagons with n vertices whose external product equals a fixed perfect square N and such that the only entry that can appear more than once is 1 *is equal to:*

$$\sum_{l=1}^{n} \left\lceil \frac{(n-1)!}{2(n-l)!} \right\rceil P_d(l, \sqrt{N}).$$

Now, we just need to investigate how to count the factorizations of $\sqrt{N}$ in l distinct factors $P_d(l, \sqrt{N})$, where $1 \leq l \leq n$.

A closed form for $P_d(l, \sqrt{N})$ is still an open problem. However, Arnold Knopfmacher and ME Mays published an article where they describe an algorithmic process to find $P_d(l, \sqrt{N})$. Moreover, they have an application in Mathematica to calculate this number.

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Consider the square arithmagons whose external product is 900. We need to consider factorizations of $30 = 2 \cdot 3 \cdot 5$.

So $P_d(3, 30) = 1$, because there is just one factorization of 30 as a product of three distinct factors.

Similarly $P_d(2, 30) = 3$, because $30 = 6 \cdot 5 = 2 \cdot 15 = 10 \cdot 3$ are the only factorizations of 30 in terms of two distinct factors.

Finally, $P_d(1, 30) = 1$ as $30 = 30$ is the only factorization of 30 in terms of a unique term and $P_d(4, 30) = 0$.

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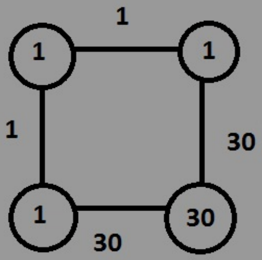
Then, our theorem states that there are

$$\sum_{l=1}^{4}\left[\frac{3!}{2(4-l)!}\right]P_d(l,30) = 1\cdot 1 + 2\cdot 3 + 3\cdot 1 = 10$$

The first term in the summation represents the arithmagon:



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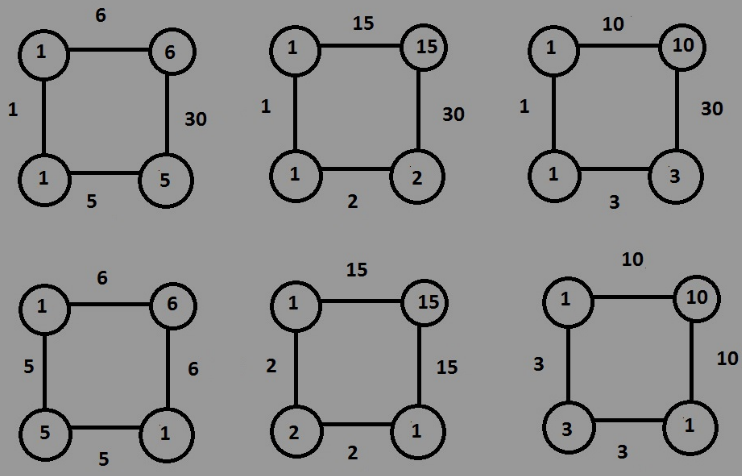

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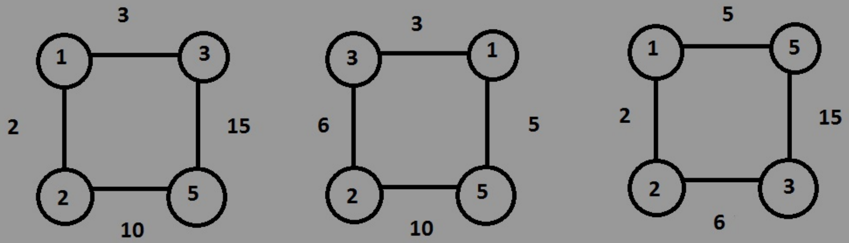

and those are our 10 arithmagons with exterior product equal to 900.

To close, we can see how counting the multiplicative arithmagons secretly solved the problem of the additive arithmagons. The trick is the same as we always use to pass from multiplication to addition. That is, using exponentiation.

It just suffices to observe that a factorization of a^N in k parts gives a partition of N in k parts and vice versa. So, this sets a bijective correspondence between the set of multiplicative n-arithmagons with exterior product equal to a^N and the set of additive n-arithmagons whose exterior sum equals N.

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That's all folks!

Thank you!